TWO-LEVEL GALERKIN MIXED FINITE ELEMENT METHOD FOR GENERALIZED DARCY-FORCHHEIMER PROBLEMS

BY

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I dedicate this dissertation to my parents Mr. Benjamin Audu Onoja and Mrs. Alice Onoja.

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THESIS ABSTRACT

NAME: Johnson Daddy Audu

TITLE OF STUDY: Two-Level Galerkin Mixed Finite Element Method For Generalized Darcy-Forchheimer Problems

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We consider the generalized Darcy-Forchheimer equation

$$\frac{\mu}{\rho}\boldsymbol{K}^{-1}\boldsymbol{u} + \frac{\beta}{\rho}|\boldsymbol{u}|^{m-1}\boldsymbol{u} + \bigtriangledown p = \boldsymbol{f} \qquad \text{in } \ \Omega, \ \ m \in (1, \ \ 2],$$

where Ω an open bounded domain of $\mathbb{R}^d$ $(d = 2, 3)$, $\boldsymbol{u}$ is the velocity, p is the pressure and $\boldsymbol{f}$ is a given function. The parameters ρ, μ, and β represent the density, viscosity and dynamic viscosity respectively while $\boldsymbol{K}$ is the permeability tensor. This equation serves as a model for many high velocity flows in porous media, most especially for gas reservoirs and hydrodynamic flows. Under some mild regularity assumptions on the boundary and interior data, we establish the well-posedness of the model. For any $m \in (1, 2]$, we consider mixed finite element approximations of the model using piecewise constant and piecewise linear mixed elements; $[L^m(\Omega)]^d$ and $[L^{\frac{m+1}{m}}(\Omega)]^d$ error estimates of the velocity and pressure respectively are of order $O(h)$, where h is the mesh size. For the case $m = 2$, we propose and analyze a two-level method for the model. In this case, a priori error estimates are obtained in $[L^2(\Omega)]^d$ and $[L^{\frac{3}{2}}(\Omega)]^d$ norms respectively. We deduce from these estimates that the coarse and fine mesh are related by $h = O(H^2)$, where h and H are the fine and coarse mesh sizes respectively. The efficiency and accuracy of the method were computationally examined. Numerical results show that the two-level Galerkin mixed finite element method is computationally cost-effective than standard Galerkin mixed finite element and the rate of convergence also agrees with the theoretical results

CHAPTER 1

INTRODUCTION

In the field of fluid dynamics, the nonlinear correction to Darcy's law has been an active area of research for many years. Theoretical, experimental, and numerical analysis [28, 47, 25] have been invested to ascertain the exact form and magnitude of the nonlinearity effect. However, until now, no single correction seems to be acceptable by all [25]. Non-Darcy effects are prevalent in fluid transport through porous media especially for high velocity flows [59]. The application of nonlinear Darcy-Forchheimer equation is seen in many fields, worthy of note includes posing as a mathematical model for many high velocity flows in porous media, most especially for gas reservoirs and hydrodynamic flows [26, 27, 46]. The generalized nonlinear Darcy-Forchheimer equation takes the form,

$$G(\rho, \boldsymbol{u}) + \bigtriangledown p = \boldsymbol{f} \quad \text{in} \quad \Omega, \tag{1.1}$$

In (2.4), p is the pressure, $\boldsymbol{u}$ and ρ are the velocity and density of the fluid, respectively. $\boldsymbol{f}$ is a vector function, usually the gradient of the depth function. Different forms of (2.4) have been considered by several researchers, for instance, see [4, 35, 20, 26, 27, 46]. The model of interest in this dissertation is the following generalized flow law [16, 38]

$$\frac{\mu}{\rho}\boldsymbol{K}^{-1}\boldsymbol{u} + \frac{\beta}{\rho}|\boldsymbol{u}|^{m-1}\boldsymbol{u} + \bigtriangledown p = \boldsymbol{f} \quad \text{in} \quad \Omega, \qquad m \in (1, 2], \tag{1.2}$$

where Ω is an open bounded domain of $\mathbb{R}^d$ ($d = 2$ or 3), with Lipschitz boundary Γ, $\boldsymbol{u} : \Omega \longrightarrow \mathbb{R}^d$ and $p : \Omega \longrightarrow \mathbb{R}$ denote the unknown velocity vector and scalar pressure field respectively while $\boldsymbol{f}$ is a given function. The permeability tensor is assumed to be uniformly positive definite and bounded while the parameters μ, ρ and β are assumed to be constants. We denote by $|\cdot|$ the Euclidean norm such that $|\boldsymbol{u}|^{m-1} = (\boldsymbol{u}\cdot\boldsymbol{u})^{\frac{m-1}{2}}$. For m=2, (1.2) reduces to the classical Darcy-Forchheimer law [28, 6, 48].

We solve (1.2) subject to the following conditions.

$$\mathbf{div}\boldsymbol{u} = b \quad \text{in} \qquad \Omega, \quad \text{and} \quad \boldsymbol{u}\cdot\boldsymbol{n} = g \qquad \text{on} \qquad \Gamma. \tag{1.3}$$

In (1.3), b and g are the interior and boundary data respectively satisfying the compatibility condition,

$$\int_\Omega b(x)dx = \int_\Gamma g(\sigma)d\sigma. \tag{1.4}$$

(1.2)-(1.4) is a reliable model for describing a single phase strong inertia flow [45, 14, 13] in simple and complex porous media.

1.1 Literature review

Problem (1.2)-(1.4) is nonlinear and of monotone type. For the case $m = 2$, the existence, uniqueness and stability of the continuous solution was accomplished by Girault and Wheeler in [31]. There in [31], the authors further studied the mixed finite element discretization using piecewise constant velocities and nonconforming piecewise linear pressures. The numerical implementation of the scheme in [31] was presented by Salas etal in [57, 44]. In contrast, approximation of the velocity and pressure using Raviart-Thomas, Brezzi-Douglas-Marini mixed finite elements were considered by Pan and Rui in [51]. A semi-discrete mixed finite element discretization of Forchheimer equation was investigated by Park in [52]. Different mixed finite elements discretizations were also compared in [51, 44]. Different numerical discretization of Darcy-Forchheimer equation were compared in [44]. Rui and Pan in [55] proposed the block-centered finite difference method, Rui, Zhao and Pan in [56] presented a block-centered finite difference method with variable nonlinear parameter. Wang and Rui in [62] constructed a stabilized method using Crouzeix-Raviart element.

However, irrespective of the method of discretization, the Darcy-Forchheimer equation give rise to system of nonlinear equations which is computationally expensive. Alternating direction iterative method of Peaceman Rashford-type was utilized in [31] to solve the resulting nonlinear system, Newton's iterative method was used in [44]. Huang, Chen and Rui in [36] applied full approximation scheme to construct a V-cycle multigrid method for the nonlinear Darcy-Forchheimer equation. Two-level method was introduced by Xu [63, 64] in his quest to reduce the computational cost of nonlinear systems of equation. The method starts by solving one small, nonlinear system of equations on a coarse mesh $\mathcal{T}_H$ with mesh size H to obtain a rough approximation followed by solving large linear system of equations on a finer mesh $\mathcal{T}_h$ of size h with $h << H$ to produce corrected solution. Many authors have since explored this technique for different nonlinear equations, for instance, Layton [39], Layton and Lenferink [41, 40], Layton and Tobiska [42], Fairag [22, 23, 24], Foster, Iliescu and wells [29], Dawson and Wheeler [18], Dawson, Wheeler and Woodward [19], Chen, Liu and Liu [15], He [33, 32], He and Li [34], Li and Rui [43].

The first attempt to solve the Darcy-Forcheimer model using two-level method was accomplished by Rui and Liu in [54]. They obtained approximate solution using two-grid block centered finite difference method. In addition, an error estimates of order $O(h^2 + H^4 + \epsilon)$ were obtained, for both velocity and pressure, ϵ is a small positive parameter. To the best of our knowledge, no research work has incorporated the two-level method with mixed finite element to solve the Darcy-Forchheimer model. Furthermore, for the case $m \in (1, 2)$, it is also worth to mention that to the best of our knowledge, no well-posedness results of the continuous solution nor numerical approximation of the model has been presented.

1.2 Contribution of the dissertation

For the case $m = 2$, the existence, uniqueness and stability of the continuous solution of Problem (1.2)-(1.4) was accomplished by Girault and Wheeler in [31]. Furthermore, mixed finite element approximations of the Problem were studied in [31, 44, 57]. In this dissertation, we achieved the following.

- **THEORETICAL ANALYSIS:**

  i. Proof of the existence and uniqueness of the solution $(\boldsymbol{u}, p)$ of (1.2)-(1.4) in $[L^{m+1}(\Omega)]^d \times (W^{1,\frac{m+1}{m}}(\Omega) \cap L_0^2(\Omega))$ and its stability under some mild regularity assumptions on b and g. The results are based on well established classical arguments for nonlinear monotone-type problems [58, 21] and on techniques used by Girault and Wheeler in [31].

  ii. Proposed and analyzed a standard mixed finite element scheme for the generalized Darcy-Forchheimer equation. The wellposenednes of the scheme was established. The following error estimates were obtained:

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{[L^m(\Omega)]^d} \leq Ch|\boldsymbol{u}|_{[W^{1,\frac{2m}{m-1}}(\Omega)]^d},$$

$$\| \bigtriangledown (p - p_h)\|_{[L^{\frac{m+1}{m}}(\Omega)]^d} \leq Ch\left(\|\boldsymbol{u}\|_{[W^{1,\frac{2m}{m-1}}(\Omega)]^d} + |p|_{W^{2,\frac{m+1}{m}}(\Omega)}\right),$$

  where the pair $(\boldsymbol{u}_h, p_h)$ solves the approximate Problem (3.9)-(3.10).

  iii. Developed and analyzed a two-level method for the mixed finite element discretization of the Darcy-Forchheimer equation (case m=2), in which the nonlinearity occur in the velocity term. We obtained $[L^2(\Omega)]^d$ and $[L^{\frac{3}{2}}(\Omega)]^d$ velocity and pressure error estimates respectively for the two-level scheme. The wellposedness of the numerical scheme was established and the error bounds for the velocity and pressure were also obtained. The procedures and estimates are based on well established analysis of two level method and mixed finite element, see [7, 63, 30]. The following order of convergence were established:

$$\|\boldsymbol{u} - \boldsymbol{u_h}\|_{[L^2(\Omega)]^d} = O(\epsilon h + h + H^2 + \epsilon),$$

$$\| \bigtriangledown (p - p_h)\|_{[L^{3/2}(\Omega)]^d} = O(\epsilon h + h + H^2 + \epsilon).$$

•NUMERICAL IMPLEMENTATION

The following MATLAB codes have been developed.

i. A two-dimensional, two-level code based on mixed finite element for two-term Darcy-Forchheimer equation.

ii. A two-dimensional standard mixed finite element for generalized Darcy-Forchheimer equation.

With these codes, the two-level scheme proved to be computationally less expensive and also conforms with other theoretical results earlier stated.

1.3 Structure of the dissertation

The dissertation is organized as follows:

1. Chapter 2 presents notations, definitions, inequalities and some established results.

2. Chapter 3 contains the wellposedness of the continuous problem, this includes existence, uniqueness and stability.

3. Chapter 4 presents the mixed finite element discretization method for solving generalized nonlinear Darcy-Forchheimer equation. It also contains the existence, uniqueness and stability of the discrete scheme and error estimates for the velocity and the pressure. The convergence analysis of the mixed finite element method is further presented.

4. The two-level method for the two-term nonlinear Darcy-Forchheimer equation is presented in chapter 5. The well-posedness and error estimates of the scheme are included.

5. The implementation of the numerical schemes and the numerical results are presented in Chapter 6.

6. The conclusion and direction for future work is contained in Chapter 7.

1.4 Preliminaries

1.4.1 Functional spaces

Let Ω be a subset of $\mathbb{R}^d$,

$$L^p(\Omega) = \left\{ u : \Omega \longrightarrow \mathbb{R} \text{ is measurable } \text{ and } \int_\Omega |u|^p dx < \infty, 1 \leq p < \infty \right\}.$$

The L^p norm of a function f is given by,

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u|^p \right)^{\frac{1}{p}},$$

$$\|u\|_{L^\infty(\Omega)} = \inf \left\{ k \geq 0 : |u(x)| \leq k \text{ for almost every } x \in \Omega \right\}.$$

Let $\alpha = (\alpha_1, \alpha_2, \alpha_3, ..., \alpha_d)$, $\alpha \in \mathbb{N}$ be multi-index with $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3 + \cdots + \alpha_d$.

Define the partial derivatives,

$$\mathcal{D}^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3} \cdots \partial x_d^{\alpha_d}} \quad .$$

For any $m \in \mathbb{N}$ and $1 \leq p \leq +\infty$, we define the Sobolev space(see [2])

$$W^{m,p} = \{ u \in L^p : \mathcal{D}^\alpha u \in L^p(\Omega), \quad |\alpha| \leq m \}.$$

Equipped with the norm,

$$\|u\|_{W^{m,p}(\Omega)} = \left(\sum_{0 \le k \le m} \|\mathcal{D}^\alpha u\|^p_{L^p(\Omega)} \right)^{1/p}, \ 1 \le p < \infty. \tag{1.5}$$

$$\|u\|_{W^{m,\infty}(\Omega)} = \max_{0 \le k \le m} \|\mathcal{D}^\alpha u\|^p_{L^\infty(\Omega)} \quad . \tag{1.6}$$

For $p = 2$, $W^{m,p}(\Omega) = H^m(\Omega)$ is a Hilbert space. For a vector-valued function $\boldsymbol{u} = (u_1, u_2, \cdots, u_d) \in [W^{\ell,p}(\Omega)]^d$, the norm

$$\|\boldsymbol{u}\|_{W^{\ell,p}(\Omega)} = \left(\sum_{i=1}^{d} \|\boldsymbol{u}_i\|^p \right)^{\frac{1}{p}}.$$

Note, $\| \cdot \|_{[W^{\ell,p}(\Omega)]^d}$ reduces to $\| \cdot \|_{[L^p(\Omega)]^d}$ when $\ell = 0$. We denote,

$$\|\boldsymbol{u}\|_{[L^p(\Omega)]^d} = \left(\int_\Omega \sum_{i=1}^{d} |u_i(x)|^p dx \right)^{\frac{1}{p}}$$

In the sequel, C is a generic constant independent of the mesh size element.

1.4.2 Some embedding theorems

Suppose Ω is bounded domain of $\mathbb{R}^d$ and of class C^1, then the following embeddings are compact [2].

$$W^{1,p}(\Omega) \hookrightarrow L^q(\Omega), \quad \forall q \in [1, p^*), p^* = \frac{pd}{d-p}, \ \text{if} \ \ p < d, \tag{1.7}$$

$$W^{1,p}(\Omega) \hookrightarrow L^q(\Omega), \quad \forall q \in [1, +\infty), \ \text{if} \ \ p = d, \tag{1.8}$$

$$W^{1,p}(\Omega) \hookrightarrow C(\bar{\Omega}), \ \text{if} \ \ p > d. \tag{1.9}$$

The following trace functions are continuous.

$$\hat{\delta} : W^{1,p}(\Omega) \longrightarrow L^q(\Gamma), \quad \forall p \leq q \leq \frac{(d-1)p}{d-1}, \text{ if } \ p < d, \tag{1.10}$$

$$\hat{\delta} : W^{1,p}(\Omega) \longrightarrow L^q(\Gamma), \quad \forall p \leq q \leq +\infty, \text{ if } \ p = d, \tag{1.11}$$

$$\hat{\delta} : W^{1,p}(\Omega) \longrightarrow L^\infty(\Gamma), \text{ if } \ p > d. \tag{1.12}$$

1.4.3 Inequalities

The following inequalities can be found in [2, 58].

$$(a+b)^p \leq 2^{p-1}(a^p + b^p), \forall a, b \in [0, \infty), \quad p \in (1, \infty), \tag{1.13}$$

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \quad a, b \in [0, \infty), \ \frac{1}{p} + \frac{1}{q} = 1, \quad p, q \in (0, \infty), \tag{1.14}$$

$$\|uv\|_1 \leq \|u\|_p \|v\|_q, \frac{1}{p} + \frac{1}{q} = 1, \quad \forall u \in L^p, \quad v \in L^q, \tag{1.15}$$

$$\|u\|_{L^q(\Omega)} \leq C\|u\|_{W^{m,p}(\Omega)}, \quad \forall u \in W^{m,p}(\Omega)^d, \quad m + \frac{d}{q} \geq \frac{d}{p}, q < \infty. \tag{1.16}$$

1.4.4 Definitions

Here, we give the meaning of some terminologies used in this dissertation.

Let X be a Banach space, $\mathcal{A} : X \longrightarrow X^*$ be a nonlinear operator, and u, v $w \in X$. $\mathcal{A}$ is said to be

- Bounded $\Leftrightarrow$ $\mathcal{A}$ maps bounded sets into bounded sets.
- Monotone $\Leftrightarrow$ $\langle \mathcal{A}(u-v), u-v \rangle \geq 0$.
- Strictly Monotone $\Leftrightarrow$ $\langle \mathcal{A}(u-v), u-v \rangle > 0 \ \ u \neq v$.

- Strongly monotone $\Leftrightarrow \langle \mathcal{A}(u-v), u-v\rangle \geq c\|u-v\|_X^2,\ c \in \mathbb{R}_+$.

- Coercive $\Leftrightarrow \lim\limits_{\|u\|_X \to \infty} \frac{\langle \mathcal{A}(u),u\rangle}{\|u\|_X}$.

- Hemicontinuous $\Leftrightarrow t \to \langle \mathcal{A}(u+tv), w\rangle$ is continous on $\mathbb{R}$.

1.4.5 Mixed variational nonlinear problem

Consider two Banach spaces, say X and M endowed with the norms $\|\cdot\|_X$ and $\|\cdot\|_M$ respectively. Suppose there is a continuous bilinear form b on $X \times M$ and a nonlinear form a defined on $X \times X$. We consider the following nonlinear problem [30, 7]

Given $(f, g) \in X^* \times M^*$, find a pair $(u, p) \in X \times M$ such that

$$\begin{cases} a(u,v) + b(v,p) = \langle f, v\rangle_{X^*,X}, \forall v \in X, \\ \qquad\qquad b(u,q) = \langle g, q\rangle_{M^* \times M}, \forall q \in M, \end{cases} \tag{1.17}$$

where $\langle\ ,\ \rangle$ represent duality pairing. We define define operators below:

$$\begin{aligned} \mathcal{A} \quad &: X \to X^*, \langle \mathcal{A}(u), v\rangle_{X^* \times X} \quad : \ = a(u,v), \forall u, v \in X, \\ \mathcal{B} \quad &: X \to M^*, \langle \mathcal{B}u, q\rangle_{M^* \times M} \quad : \ = b(u,q), q \in M, \\ \mathcal{B}^* \quad &: M \to X^*, \langle \mathcal{B}^*(p), v\rangle_{X^* \times X} \quad : \ = b(v,p), v \in X. \end{aligned}$$

Now, rewrite Problem (1.17) as follows: find $(u, p) \in X \times M$ such that,

$$\begin{cases} \mathcal{A}u + \mathcal{B}^*(p) = f \quad \text{in} \quad X^*, \\ \qquad\qquad \mathcal{B}u = g \quad \text{in } M^*. \end{cases} \tag{1.18}$$

Now, denote by

$$\begin{aligned} V := Ker\mathcal{B} &= \{u \in X, \quad \mathcal{B}u = 0\} \\ &= \{u \in X, \quad b(u,q) = 0, \forall q \in M\} \end{aligned} \tag{1.19}$$

Then the problem associated with (1.18) is to find $u \in V$ satisfying,

$$\mathcal{A}u = f \quad \text{in} \quad X^*. \tag{1.20}$$

The well-posedness of Problem (1.20) is guaranteed by the following theorem.

Theorem 1.1 *[12, 21, Browder 1963, Minty 1963, pp.557]*

Let $\mathcal{A} : X \longrightarrow X^*$ be monotone, coercive, and hemicontinous operator on a real, separable and reflexive Banach space. Then, the nonlinear stationary problem,

$$\mathcal{A}u = f \quad in \quad X^*, \tag{1.21}$$

has a solution $u \in X$. If $\mathcal{A}$ is strictly monotone, then the solution is unique.

Theorem 1.2 *[7]*

Suppose that the bilinear form b satisfies the inf-sup condition:

$$\sup_{v \in X \setminus \{0\}} \frac{b(v,q)}{\|v\|_X} \geq \beta \|q\|_M, \forall q \in M, \tag{1.22}$$

for some $\beta > 0$ Then, for each solution u of the problem (1.20), there exists a *unique $p \in M$ such that the pair (u,p) satisfies problem* (1.18).

Theorem 1.3 (Necas[11]) *Let X be Banach space and M be a reflexive Banach space. Let $a(\cdot,\cdot)$ be a bilinear continuous form on $X \times M$. Suppose $f \in X^*$ then, the problem of finding $u \in X$ such that $\forall v \in M, \quad a(u,v) = \langle f, v\rangle$ is well posed if and only if*

$$\exists \alpha > 0, \ \inf_{w\in X} \sup_{v\in M} \frac{a(w,v)}{\|w\|_X \|v\|_M} \geq \alpha, \tag{1.23}$$

and

$$\forall v \in M, \text{if } \ a(w,v) = 0, \quad \forall w \in X, \ \text{then } \ v = 0. \tag{1.24}$$

Furthermore, $\|u\|_X \leq \frac{1}{\alpha}\|f\|_{X^*}$.

Lemma 1.1 *[30] Let $(X, \|\cdot\|_X)$ and $(M, \|\cdot\|_M)$ be two reflexive Banach spaces. Let $(X^*, \|\cdot\|_{X^*})$, $(M^*, \|\cdot\|_{M^*})$ be their corresponding dual. Let $\mathcal{B}$: $X \longrightarrow M^*$ be a linear continuous operator and $\mathcal{B}^* : M \longrightarrow X^*$ be the dual of $\mathcal{B}$. Let $V = ker(\mathcal{B})$ be kernel of $\mathcal{B}$. Denote by $V^0 \subset X$ the polar subspace of V, $V^0 = \{x^* \in X^* | \langle x^*, v\rangle = 0, \forall v \in X\}$ and $\dot{\mathcal{B}} : X/V \longrightarrow M^*$ the quotient operator associated with $\mathcal{B}$. Then the following properties are equivalent:*

i.

$$\inf_{q\in M\setminus\{0\}} \sup_{u\in X\setminus\{0\}} \frac{\langle \mathcal{B}u, q\rangle}{\|q\|_M \|u\|_X} \geq \alpha, \quad \alpha > 0 \tag{1.25}$$

ii. *$\mathcal{B}^*$ is an isomorphism from M onto V^0 and*

$$\|\mathcal{B}^* q\|_{X^*} \geq \alpha \|q\|_M, \quad \alpha > 0, \quad \forall q \in M. \tag{1.26}$$

iii. $\dot{\mathcal{B}}$ *is an isomorphism from* X/V *onto* M^* *and*

$$\|\dot{M}\dot{u}\|_{M^*} \geq \alpha\|\dot{u}\|_{X/V} \quad , \quad \alpha > 0, \quad \forall \dot{u} \in X/V. \tag{1.27}$$

Corollary 1.4 *[30, 7] The following assertions are equivalent*

i. *The inf-sup condition in* (1.25).

ii. $\mathcal{B}^* : M \longrightarrow X^*$ *is injective and* $\mathcal{B}^*$ *has a closed range.*

iii. $\mathcal{B}$ *is surjective.*

Theorem 1.5 (The mean value theorem for vector-valued functions [60]) *Let* $\psi : D \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ *be a differentiable vector valued function on a convex open set* D. *For any arbitrary points* $\boldsymbol{v}, \boldsymbol{w} \in D$,

$$|\psi(\boldsymbol{v}) - \psi(\boldsymbol{w})| < \sup_{0 \leq < t \leq 1} \|\psi'(\boldsymbol{w} + t(\boldsymbol{v} - \boldsymbol{w})\||\boldsymbol{v} - \boldsymbol{w}| \tag{1.28}$$

ψ' *is the Jacobian matrix* $\partial\psi_i/\partial\psi_j$, $i = i, 2, 3 \cdots m, j = 1, 2, 3 \cdots n$. $\| \ \|$ *is the norm on the set of linear maps from* $\mathbb{R}^n$ *to* $\mathbb{R}^m$ *and* $| \ |$ *is the appropriate norm* $\mathbb{R}^n$.

1.4.6 Mixed finite element method

In application, many mathematical models give rise to a systems of partial differential equations that consist of several physical quantities that need to be simultaneously approximated. Mixed finite element methods is a kind of finite element method in which two or more finite element spaces are required to approximate atleast two different quantities or variables. It has been successfully utilized to approximate different kind of partial differential equations. These include Stokes equation [7] and Navier-Stokes [30] Here, we present a brief introduction for approximating two different variables. Let $X_h \subset X$ and $M_h \subset M$ be two finite dimensional subspaces of two infinite dimensional functional spaces X and M respectively. The index h refers to a mesh size from which these approximations are derived. Now, consider the restriction of the forms a and b in to X_h and M_h respectively. Then, a corresponding approximation of Problem (1.17) is to look for a couple $(u_h, p_h) \in X_h \times M_h$ satisfying,

$$\begin{cases} a(u_h, v_h) + b(v_h, q_h) = \langle f, v_h \rangle_{X',X}, \forall v_h \in X_h, \\ \qquad\qquad b(u_h, q_h) = \langle g, q_h \rangle_{M' \times M}, \forall q_h \in M_h. \end{cases} \tag{1.29}$$

Now, define

$$V_h := \{u \in X_h, b(u_h, q_h) = 0, \quad \forall q_h \in M_h\}.$$

Theorem 1.6 *[5, 11, Brezzi-Babuska Theory] The well-posedness of* (1.29) *is guaranteed by the following inf-sup conditions:*

- *An inf-sup condition relating the spaces X_h and M_h,*

$$\inf_{0 \neq q_h \in M_h,} \sup_{0 \neq v_h \in X_h} \frac{b(v_h, q_h)}{\|v_h\|_X \|q_h\|_M} \geq \beta, \tag{1.30}$$

  for some $\beta > 0$.

- *An inf-sup in the Kernel condition,*

$$\inf_{0 \neq u_h \in V_h,} \sup_{0 \neq v_h \in V_h} \frac{a(u_h, v_h)}{\|u_h\|_X \|v_h\|_X} \geq \gamma, \tag{1.31}$$

for some $\gamma > 0$.

CHAPTER 2

WELL-POSEDNESS OF GENERALIZED DARCY-FORCHHEIMER EQUATION

2.1 Variational formulation

The problem (1.2)-(1.4) is nonlinear of monotone type. We define the following spaces:

$$L_0^2(\Omega) = \{v : v \in L^2(\Omega), \int_\Omega v(x)dx = 0\},$$

$$X = [L^{m+1}(\Omega)]^d,$$

$$M = W^{1,\frac{m+1}{m}}(\Omega) \cap L_0^2(\Omega).$$

The zero mean condition is required to guarantee the uniqueness of the pressure p, since p plus any constant $c \in \mathbb{R}$ also satisfy (1.2)-(1.4). We consider the following variational formulation:

For any $\boldsymbol{f} \in X^*$, find a pair of functions $(\boldsymbol{u}, p) \in X \times M$ such that,

$$\frac{\mu}{\rho}\int_\Omega (K^{-1}\boldsymbol{u}) \cdot \boldsymbol{\varphi} dx + \frac{\beta}{\rho}\int_\Omega |\boldsymbol{u}|^{m-1}\boldsymbol{u} \cdot \boldsymbol{\varphi} d\boldsymbol{x} + \int_\Omega \bigtriangledown p \cdot \boldsymbol{\varphi} d\boldsymbol{x} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi} d\boldsymbol{x}, \forall \boldsymbol{\varphi} \in X \quad (2.1)$$

$$\int_\Omega \bigtriangledown q \cdot \boldsymbol{u} d\boldsymbol{x} = -\int_\Omega bq\boldsymbol{d}x + \int_\Gamma gq\boldsymbol{d}\sigma, \ \forall q \in M, \quad (2.2)$$

with $b \in L^{\frac{d(m+1)}{d+(m+1)}}(\Omega)$ and $g \in L^{\frac{(d-1)(m+1)}{d}}(\Gamma)$ satisfying (1.4).

Problems (2.1)-(2.2) and (1.2)-(1.4) are equivalent. To see this, we multiply equation (1.2) by $\boldsymbol{\varphi} \in X$ and integrate over Ω, and also subject equation (1.3) to the Green's formula.

$$\int_\Omega \boldsymbol{v} \cdot \bigtriangledown q d\boldsymbol{x} = -\int_\Omega q\,\mathbf{div}\boldsymbol{v} d\boldsymbol{x} + < q, \boldsymbol{v} \cdot n >_\Gamma, \forall q \in M, \forall \boldsymbol{v} \in H, \quad (2.3)$$

with

$$H = \left\{ \boldsymbol{v} \in [L^{m+1}(\Omega)]^d; \mathbf{div}\boldsymbol{v} \in L^{\frac{d(m+1)}{d+(m+1)}}(\Omega) \right\}.$$

Define the following operators:

$$\begin{aligned}
\mathcal{A} \quad &: X \to X^*, \quad \langle \mathcal{A}(\boldsymbol{u}), \boldsymbol{\varphi} \rangle_{X^* \times X} &&:= a(\boldsymbol{u}, \boldsymbol{\varphi}), \quad \forall \boldsymbol{\varphi} \in X, \\
\boldsymbol{B} \quad &: X \to M^*, \quad \langle \boldsymbol{B}\boldsymbol{u}, q \rangle_{M^* \times M} &&:= b(\boldsymbol{u}, q), \quad q \in M, \\
\boldsymbol{B}^* \quad &: M \to X^*, \quad \langle \boldsymbol{B}^*(p), \boldsymbol{\varphi} \rangle_{X^* \times X} &&:= b(\boldsymbol{\varphi}, p), \quad \boldsymbol{\varphi} \in X.
\end{aligned}$$

With these operators, an equivalent form of (2.1)-(2.2) is as follows:

Given $(\boldsymbol{f}, g) \in X^* \times M^*$, we want to find a pair $(\boldsymbol{u}, p) \in X \times M$ solution of

$$\begin{cases} a(\boldsymbol{u}, \boldsymbol{\varphi}) + b(\boldsymbol{\varphi}, p) = \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle_{X^*, X}, \forall \boldsymbol{\varphi} \in X, \\ \\ b(\boldsymbol{u}, q) = \langle g, q \rangle_{M^* \times M}, \forall q \in M. \end{cases} \tag{2.4}$$

Note that Problem (2.4) can also be written as:

$$\begin{cases} \mathcal{A}(\boldsymbol{u}) + \boldsymbol{B}^*(p) = \boldsymbol{f} \quad \text{in} \quad X^*, \\ \\ \boldsymbol{B} u = g \quad \text{in} \quad M^*, \end{cases} \tag{2.5}$$

where,

$$\mathcal{A}(\boldsymbol{u}) = \frac{\mu}{\rho} \boldsymbol{K}^{-1} \boldsymbol{u} + \frac{\beta}{\rho} |\boldsymbol{u}|^{m-1} \boldsymbol{u}, \tag{2.6}$$

$$\langle \boldsymbol{B u}, q \rangle = \int_\Omega \nabla q \cdot \boldsymbol{u} d\boldsymbol{x}, \tag{2.7}$$

$$\langle \boldsymbol{B}^*(p), \boldsymbol{u} \rangle = \int_\Omega \nabla p \cdot \boldsymbol{u} d\boldsymbol{x}, \tag{2.8}$$

$$F(q) = \langle g, q \rangle_{M^* \times M} = -\int_\Omega b q \boldsymbol{d} x + \int_\Gamma g q \boldsymbol{d} \sigma. \tag{2.9}$$

To establish the wellposedness of (2.1)-(2.2), we employ Theorems 1.1 and 1.2. To begin, we prove the "inf-sup" condition below.

Lemma 2.1 *The following inf-sup condition holds:*

$$\inf_{q \in M \setminus \{0\}} \sup_{\boldsymbol{u} \in X \setminus \{0\}} \frac{b(\boldsymbol{u}, q)}{\|q\|_M \|\boldsymbol{u}\|_X} = 1. \tag{2.10}$$

Proof. By representation of dual norm,

$$\|\boldsymbol{v}\|_{[L^{\frac{m+1}{m}}(\Omega)]^d} = \sup_{\boldsymbol{u}\in[L^{m+1}(\Omega)]^d} \frac{\int_\Omega \boldsymbol{v}\cdot\boldsymbol{u}d\boldsymbol{x}}{\|\boldsymbol{u}\|_{[L^{m+1}(\Omega)]^d}}, \quad \forall \boldsymbol{v}\in[L^{\frac{m+1}{m}}(\Omega)]^d. \tag{2.11}$$

By setting $\boldsymbol{v} = \bigtriangledown q$, we get

$$\|\bigtriangledown q\|_{[L^{\frac{m+1}{m}}(\Omega)]^d} = \sup_{\|\boldsymbol{u}\|_{[L^{m+1}(\Omega)]^d}} \frac{\int_\Omega \bigtriangledown q\cdot\boldsymbol{u}d\boldsymbol{x}}{\|u\|_{[L^{m+1}(\Omega)]^d}}, \qquad \forall q\in M. \tag{2.12}$$

Since q belongs to the space of zero mean, we have the following

$$\|\bigtriangledown q\|_{[L^{\frac{m+1}{m}}(\Omega)]^d} = \|q\|_M, \quad \forall q\in M.$$

Equivalently,

$$\inf_{q\in M\setminus\{0\}} \frac{\|\bigtriangledown q\|_{[L^{\frac{m+1}{m}}(\Omega)]^d}}{\|q\|_M} = 1. \tag{2.13}$$

Now substituting (2.12) in (2.13) yields the required result. ∎

Proposition 2.1 *For each* $b\in L^{\frac{d(m+1)}{d+(m+1)}}(\Omega)$ *and* $g\in L^{\frac{(d-1)(m+1)}{d}}(\Gamma)$, *satisfying* (1.4) *there is a unique* $\boldsymbol{u}_\ell\in[L^{m+1}(\Omega)]^d/V$ *satisfying*

$$\int_\Omega \boldsymbol{u}_\ell\cdot\bigtriangledown q d\boldsymbol{x} = -\int_\Omega bq d\boldsymbol{x} + \int_\Gamma gq d\sigma, \quad \forall q\in M. \tag{2.14}$$

Consequently,

$$\|\boldsymbol{u}_\ell\|_{L^{m+1}/V} \le C\left(\|b\|_{L^{\frac{d(m+1)}{d+(m+1)}}(\Omega)} + \|g\|_{L^{\frac{(d-1)(m+1)}{d}}(\Gamma)}\right), \tag{2.15}$$

where C is a constant depending on Ω only.

Proof. Denote,

$$\begin{aligned} V &:= Ker\boldsymbol{B} = \{\boldsymbol{u} \in X, \boldsymbol{B}\boldsymbol{u} = 0\} \\ &= \left\{\boldsymbol{u} \in X,\ b(\boldsymbol{u}, q) = \int_\Omega \triangledown q \cdot \boldsymbol{u} d\boldsymbol{x} = 0,\ \forall q \in M\right\}. \end{aligned} \tag{2.16}$$

The second equation in Formulation (2.4) can be written as:

$$b(\boldsymbol{u}, q) = F(q), \forall q \in M, \tag{2.17}$$

where $F(q) = -\int_\Omega bq\boldsymbol{d}x + \int_\Gamma gq\boldsymbol{d}\sigma$. Now let us estimate the RHS of (2.17).

$$\begin{aligned} |F(q)| &= \left| -\int_\Omega bqd\boldsymbol{x} + \int_\Gamma gqd\sigma \right|, \qquad (2.18) \\ &\leq \|b\|_{L^{\frac{d(m+1)}{d+(m+1)}}(\Omega)} \|q\|_{L^{\frac{d(m+1)}{m(d-1)-1}}(\Omega)} + \|g\|_{L^{\frac{(d-1)(m+1)}{d}}(\Gamma)} \|q\|_{L^{\frac{(d-1)(m+1)}{m(d-1)-1}}(\Gamma)}, \end{aligned}$$

So,

$$|F(q)| \leq \left(\|b\|_{L^{\frac{d(m+1)}{d+(m+1)}}(\Omega)} + \|g\|_{L^{\frac{(d-1)(m+1)}{d}}(\Gamma)} \right) \left(\|q\|_{L^{\frac{d(m+1)}{m(d-1)-1}}(\Omega)} + \|q\|_{L^{\frac{(d-1)(m+1)}{m(d-1)-1}}(\Gamma)} \right). \tag{2.19}$$

By the trace theorem and Sobolev embeddings in (1.7) and (1.10), we obtain

$$\begin{aligned}
&|F(q)| \\
&\leq \left(\|b\|_{L^{\frac{d(m+1)}{d+(m+1)}}(\Omega)} + \|g\|_{L^{\frac{(d-1)(m+1)}{d}}(\Gamma)} \right) \left(c_1 \|q\|_{W^{1,\frac{m+1}{m}}(\Omega)} + c_2 \|q\|_{W^{1,\frac{m+1}{m}}(\Omega)} \right) \\
&\leq C \left(\|b\|_{L^{\frac{d(m+1)}{d+(m+1)}}(\Omega)} + \|g\|_{L^{\frac{(d-1)(m+1)}{d}}(\Gamma)} \right) \|q\|_{W^{1,\frac{m+1}{m}}(\Omega)}
\end{aligned} \tag{2.20}$$

The map $q \mapsto -\int_\Omega bq d\boldsymbol{x} + \int_\Gamma gq d\sigma$ is a bounded linear map, so belongs to M^*. Therefore, the inf-sup condition in Lemma (2.1) and the equivalence statements in Lemma 1.1, there is a unique $\boldsymbol{u}_\ell \in [L^{m+1}(\Omega)]^d/V$ satisfying (2.14). Hence,

$$b(\boldsymbol{u}_\ell, q) \leq C \left(\|b\|_{L^{\frac{d(m+1)}{d+(m+1)}}(\Omega)} + \|g\|_{L^{\frac{(d-1)(m+1)}{d}}(\Gamma)} \right) \| \nabla q\|_{[L^{\frac{m+1}{m}}(\Omega)]^d} \tag{2.21}$$

Applying Lemma 2.1 in (2.21), (4.30) is established.

∎

Based on Proposition 2.1, we split the solution $\boldsymbol{u}$ into $\boldsymbol{u}_0 + \boldsymbol{u}_\ell$, where $\boldsymbol{u}_0 \in V$ and $\boldsymbol{u}_\ell \in [L^{m+1}(\Omega)]^d$. An equivalence variational formulation of (2.1)-(2.2) is as follows: For any $f \in X^*$, find $\boldsymbol{u}_0 \in V$ such that

$$\int_\Omega \mathcal{A}(\boldsymbol{u}_0 + \boldsymbol{u}_\ell) \cdot \boldsymbol{\varphi} d\boldsymbol{x} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi} d\boldsymbol{x}, \quad \forall \boldsymbol{\varphi} \in V, \tag{2.22}$$

Proposition 2.2 *Problem* (2.1)-(2.2) *is equivalent to Problem* (2.22).

Proof. Suppose that $(\boldsymbol{u}, p)$, with $\boldsymbol{u} \in X$, $p \in M$, is a solution of (2.1)-(2.2). Then, we write $\boldsymbol{u} = \boldsymbol{u}_0 + \boldsymbol{u}_\ell$, where $\boldsymbol{u}_\ell \in [L^{m+1}(\Omega)]^d/V$ solves (4.30). It follows that $\boldsymbol{u}_0$ satisfies (2.22).

Conversely, take $\boldsymbol{u}_0$ to be a solution of (2.22); then,

$$\int_\Omega (\boldsymbol{f} - \mathcal{A}(\boldsymbol{u}_0 + \boldsymbol{u}_\ell)) \cdot \boldsymbol{\varphi} d\boldsymbol{x} = 0, \forall \boldsymbol{\varphi} \in V.$$

Equivalently, $\boldsymbol{f} - \mathcal{A}(\boldsymbol{u}_0 + \boldsymbol{u}_\ell) \in \mathrm{V}^0$, where

$$V^0 = \left\{ \boldsymbol{v} \in [L^{\frac{m+1}{m}}(\Omega)]^d; \forall \boldsymbol{w} \in V, \int_\Omega \boldsymbol{v} \cdot \boldsymbol{w} d\boldsymbol{x} = 0 \right\} = (KerB)^0.$$

It follows from closed range Theorem [10], that $\boldsymbol{f} - \mathcal{A}(\boldsymbol{u}_0 + \boldsymbol{u}_\ell) \in Image(\boldsymbol{B}^*)$.

Thanks to the inf-sup condition in Lemma 2.1 and the Isomorphism in Lemma 1.1 there exists a unique $p \in M$ such that

$$\boldsymbol{B}^* p = \boldsymbol{f} - \mathcal{A}(\boldsymbol{u}_0 + \boldsymbol{u}_\ell).$$

Consequently,

$$\bigtriangledown p = \boldsymbol{f} - \mathcal{A}(\boldsymbol{u}_0 + \boldsymbol{u}_\ell). \tag{2.23}$$

Hence,

$$\frac{\mu}{\rho} \int_\Omega \mathcal{A}(\boldsymbol{u}_0 + \boldsymbol{u}_\ell) \cdot \boldsymbol{\varphi} d\boldsymbol{x} + \int_\Omega \bigtriangledown p \cdot \boldsymbol{\varphi} d\boldsymbol{x} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi} d\boldsymbol{x}, \quad \forall \boldsymbol{\varphi} \in X.$$

Since $\boldsymbol{u} = \boldsymbol{u_0} + \boldsymbol{u}_\ell$, (2.1) is satisfied. Furthermore, since $\boldsymbol{u}_0 \in V$, we have

$$\begin{aligned} \boldsymbol{B}\boldsymbol{u} &= \boldsymbol{B}\boldsymbol{u}_0 + \boldsymbol{B}\boldsymbol{u}_\ell \\ &= \boldsymbol{B}\boldsymbol{u}_\ell = g. \end{aligned} \tag{2.24}$$

This implies that (2.2) is satisfied. ▮

Leveraging on the equivalence in Proposition (2.2), Problem (2.22) will be the focus of our analysis.

Lemma 2.2 *For any pair* $(\boldsymbol{v}, \boldsymbol{w}) \in \mathbb{R}^n \times \mathbb{R}^n$

$$\left| |\boldsymbol{v}|^{m-1}\boldsymbol{v} - |\boldsymbol{w}|^{m-1}\boldsymbol{w} \right| \leq m2^{m-2} \left[|\boldsymbol{v}|^{m-1} + |\boldsymbol{w}|^{m-1} \right] |v - w|, m \geq 1 \tag{2.25}$$

Proof. Let $(\boldsymbol{v}, \boldsymbol{w}) \in \mathbb{R}^n \times \mathbb{R}^n$, $\psi(\boldsymbol{v}) := |\boldsymbol{v}|^{m-1}\boldsymbol{v}, \quad \psi(\boldsymbol{w}) := |\boldsymbol{w}|^{m-1}\boldsymbol{w}$

For n=2,

$$\psi(\boldsymbol{v}) = (\psi_1(v), \psi_2(v)) = (|\boldsymbol{v}|^{m-1}v_1, |\boldsymbol{v}|^{m-1}v_2),$$

$$\psi'(\boldsymbol{v}) = \begin{bmatrix} \frac{\partial \psi_1(v)}{\partial v_1} & \frac{\partial \psi_1(v)}{\partial v_2} \\ \frac{\partial \psi_2(v)}{\partial v_1} & \frac{\partial \psi_2(v)}{\partial v_2} \end{bmatrix},$$

$$\psi'(\boldsymbol{v}) = \begin{bmatrix} (m-1)|\boldsymbol{v}|^{m-3}v_1^2 + |\boldsymbol{v}|^{m-1} & (m-1)|\boldsymbol{v}|^{m-3}v_1v_2 \\ (m-1)|\boldsymbol{v}|^{m-3}v_1v_2 & (m-1)|\boldsymbol{v}|^{m-3}v_2^2 + |\boldsymbol{v}|^{m-1} \end{bmatrix}.$$

The Jacobian matrix $\psi'(\boldsymbol{v})$ is a 2×2 symmetric matrix, so its norm $\|\psi'(v)\|$ is the largest of the eigenvalue (spectral radius). The eigenvalues are computed by the formula as follows:

$$\lambda_{1,2} = \frac{1}{2}\left[(a_{11}+a_{22}) \pm \sqrt{4a_{12}a_{21}+(a_{11}-a_{22})^2}\right].$$

We estimate as follows:

$$\begin{aligned}
\sqrt{4a_{12}a_{21}+(a_{11}-a_{22})^2} &= \sqrt{4(m-1)^2|\boldsymbol{v}|^{2(m-3)}v_1^2v_2^2+(m-1)^2|\boldsymbol{v}|^{2(m-3)}(v_1^2-v_2^2)^2}, \\
&= (m-1)|\boldsymbol{v}|^{(m-3)}\sqrt{4v_1^2v_2^2+(v_1^2-v_2^2)^2}, \\
&= (m-1)|\boldsymbol{v}|^{(m-3)}\sqrt{(v_1^2+v_2^2)^2}, \\
&= (m-1)|\boldsymbol{v}|^{(m-3)}|\boldsymbol{v}|^2, \\
&= (m-1)|\boldsymbol{v}|^{m-1}. \qquad (2.26)
\end{aligned}$$

Then, $\lambda_{1,2} = \frac{1}{2}(m-1)|\boldsymbol{v}|^{m-1} + 2|\boldsymbol{v}|^{m-1} \pm (m-1)|\boldsymbol{v}|^{m-1}$.

The largest of the eigenvalue is given by,

$$\lambda_1 = \frac{1}{2}\left[(m-1)|\boldsymbol{v}|^{m-1} + 2|\boldsymbol{v}|^{m-1} + (m-1)|\boldsymbol{v}|^{m-1}\right] = m|\boldsymbol{v}|^{m-1}\ .$$

Therefore, the norm of the Jacobian matrix is given by $\|\psi'(v)\| = m|\boldsymbol{v}|^{m-1}$.

Application of Theorem 1.5 yields

$$
\begin{aligned}
|\psi(\boldsymbol{v}) - \psi(\boldsymbol{w})| &\leq \sup_{0\leq t\leq 1} ||\psi\prime(\boldsymbol{w} + t(\boldsymbol{v} - \boldsymbol{w}))|||\boldsymbol{v} - \boldsymbol{w}| \\
&\leq \sup_{0\leq t\leq 1} m|\boldsymbol{w} + t(\boldsymbol{v} - \boldsymbol{w})|^{m-1}|\boldsymbol{v} - \boldsymbol{w}| \\
&\leq m \sup_{0\leq t\leq 1} \left[|\boldsymbol{w}(1-t)| + t|\boldsymbol{v}|\right]^{m-1} |\boldsymbol{v} - \boldsymbol{w}| \\
&\leq m \left[|\boldsymbol{w}| + |\boldsymbol{v}|\right]^{m-1} |\boldsymbol{v} - \boldsymbol{w}|
\end{aligned}
\tag{2.27}
$$

So,

$$
|\psi(\boldsymbol{v}) - \psi(\boldsymbol{w})| \leq m2^{m-2} \left[|\boldsymbol{v}|^{m-1} + |\boldsymbol{w}|^{m-1}\right] |\boldsymbol{v} - \boldsymbol{w}| \tag{2.28}
$$

The last line above follows from (1.13). ∎

2.2 Existence and uniqueness

To prove the existence and uniqueness of solution for (2.22), it suffices to demonstrate that the map $\mathcal{A}$ defined in (2.6) satisfies the following properties in $[L^{m+1}(\Omega)]^d$; boundednes, strict monotonicity, coercivity and hemi-continuity [21, 58]. We let the least eigenvalue of K be λ_s. Therefore,

$$
K(\boldsymbol{x})\boldsymbol{u}\cdot\boldsymbol{u} \geq \lambda_s|\boldsymbol{u}|^2, \quad \text{for all } \ \boldsymbol{u}\in\mathbb{R}^d, \ \ \textit{and} \ \text{ for all } \ \boldsymbol{x}\in\Omega. \tag{2.29}
$$

Lemma 2.3 *The operator* $\mathcal{A} : [L^{m+1}(\Omega)]^d \to [L^{\frac{m+1}{m}}(\Omega)]^d$ *satisfies the following* *bounds:*

$\forall \boldsymbol{v},\ \forall \boldsymbol{w} \in [L^{m+1}(\Omega)]^d$,

$$\|\mathcal{A}(\boldsymbol{v})\|_{[L^{\frac{m+1}{m}}(\Omega)]^d} \leq \frac{\mu}{\rho}\|K^{-1}\|\|\boldsymbol{v}\|_{[L^{\frac{m+1}{m}}(\Omega)]^d} + \frac{\beta}{\rho}\|\boldsymbol{v}\|^m_{[L^{m+1}(\Omega)]^d}. \tag{2.30}$$

and

$$|\mathcal{A}(\boldsymbol{v}) - \mathcal{A}(\boldsymbol{w})| \leq \left(\frac{\mu}{\rho}\|K^{-1}\|_{L^\infty(\Omega)} + \frac{\beta}{\rho}m2^{m-2}\left[|\boldsymbol{v}|^{m-1} + |\boldsymbol{w}|^{m-1}\right]\right)|\boldsymbol{v} - \boldsymbol{w}|,\ \ a.\ \ e. \tag{2.31}$$

Proof. Let $\boldsymbol{u}, \boldsymbol{v} \in [L^{m+1}(\Omega)]^d$.

$\boldsymbol{v} \in [L^{m+1}(\Omega)]^d \implies \mathcal{A}(\boldsymbol{v}) \in L^{\frac{m+1}{m}}(\Omega)$. Therefore,

$$\begin{aligned}\left|\langle \mathcal{A}(\boldsymbol{v}), \boldsymbol{u}\rangle_{[L^{\frac{m+1}{m}}(\Omega)]^d\times[L^{m+1}(\Omega)]^d}\right| &= \left|\int_\Omega \left[\frac{\mu}{\rho}K^{-1}\boldsymbol{v} + \frac{\beta}{\rho}|\boldsymbol{v}|^{m-1}\boldsymbol{v}\right]\boldsymbol{u}d\boldsymbol{x}\right|,\\ &\leq \frac{\mu}{\rho}\|K^{-1}\|_{L^\infty(\Omega)}\int_\Omega |\boldsymbol{u}\boldsymbol{v}|d\boldsymbol{x} + \frac{\beta}{\rho}\int_\Omega |\boldsymbol{v}|^m|\boldsymbol{u}|d\boldsymbol{x},\\ &\leq \frac{\mu}{\rho}\|K^{-1}\|_{L^\infty(\Omega)}\|\boldsymbol{u}\|_{[L^{m+1}(\Omega)]^d}\|\boldsymbol{v}\|_{[L^{\frac{m+1}{m}}(\Omega)]^d}\\ &\quad + \frac{\beta}{\rho}\|\boldsymbol{u}\|_{[L^{m+1}(\Omega)]^d}\|\boldsymbol{v}\|^m_{[L^{m+1}(\Omega)]^d}.\end{aligned} \tag{2.32}$$

Hence,

$$\left|\langle \mathcal{A}(\boldsymbol{v}), \boldsymbol{u}\rangle_{[L^{\frac{m+1}{m}}(\Omega)]^d\times[L^{m+1}(\Omega)]^d}\right| \leq \left[\frac{\mu}{\rho}\|K^{-1}\|_{L^\infty(\Omega)}\|\boldsymbol{v}\|_{[L^{\frac{m+1}{m}}(\Omega)]^d} + \frac{\beta}{\rho}\|\boldsymbol{v}\|^m_{[L^{m+1}(\Omega)]^d}\right]\|\boldsymbol{u}\|_{[L^{m+1}(\Omega)]^d}.$$

We deduce (2.30) from the last line above. The estimate in (2.31) follows from

(2.28). ∎

Lemma 2.4 *The map $\boldsymbol{u} \mapsto \mathcal{A}(\boldsymbol{u}+\boldsymbol{u}_\ell)$ defined in* (2.6) *is strongly monotone.*

$\forall \boldsymbol{u}, \boldsymbol{v} \in [L^{m+1}(\Omega)]^d$,

$$\int_\Omega (\mathcal{A}(\boldsymbol{u}+\boldsymbol{u}_\ell) - \mathcal{A}(\boldsymbol{v}+\boldsymbol{v}_\ell)) \cdot (\boldsymbol{u}-\boldsymbol{v}) d\boldsymbol{x} \geq \frac{\mu}{\rho} \lambda_s \|\boldsymbol{u}-\boldsymbol{v}\|^2_{[L^m(\Omega)]^d} \ . \tag{2.33}$$

Proof. Define $F : [L^{m+1}(\Omega)]^d \longrightarrow \mathbb{R}$, by

$$F(\boldsymbol{v}) = \frac{\mu}{2\rho} \int_\Omega K^{-1} \boldsymbol{v} \cdot \boldsymbol{v} d\boldsymbol{x} + \frac{\beta}{\rho(m+1)} \int_\Omega |\boldsymbol{v}|^{m+1} d\boldsymbol{x}, \quad \forall \boldsymbol{v} \in [L^{m+1}(\Omega)]^d.$$

$\forall \boldsymbol{v}, \boldsymbol{w} \in [L^{m+1}(\Omega)]^d$, $h \in \mathbb{R}$,

$$\begin{aligned}
F(\boldsymbol{v}+h\boldsymbol{w}) - F(\boldsymbol{v}) &= \frac{h\mu}{\rho} \int_\Omega K^{-1} \boldsymbol{v} \cdot \boldsymbol{w} d\boldsymbol{x} + \frac{h^2\mu}{2\rho} \int_\Omega K^{-1} \boldsymbol{w} \cdot \boldsymbol{w} d\boldsymbol{x} \\
&\quad + \frac{\beta}{\rho(m+1)} \int_\Omega \left(|\boldsymbol{v}+h\boldsymbol{w}|^{m+1} - |\boldsymbol{v}|^{m+1} \right) d\boldsymbol{x}, \\
\frac{F(\boldsymbol{v}+h\boldsymbol{w}) - F(\boldsymbol{v})}{h} &= \frac{\mu}{\rho} \int_\Omega (K^{-1} \boldsymbol{v}) \cdot \boldsymbol{w} d\boldsymbol{x} + \frac{h\mu}{2\rho} \int_\Omega K^{-1} \boldsymbol{w} \cdot \boldsymbol{w} d\boldsymbol{x} \\
&\quad + \frac{\beta}{\rho(m+1)} \int_\Omega \frac{(|\boldsymbol{v}+h\boldsymbol{w}|^{m+1} - |\boldsymbol{v}|^{m+1})}{h} d\boldsymbol{x} \ .
\end{aligned}$$

Therefore,

$$\begin{aligned}
\lim_{h \to 0} \frac{F(\boldsymbol{v}+h\boldsymbol{w}) - F(\boldsymbol{v})}{h} &= \frac{\mu}{\rho} \int_\Omega K^{-1} \boldsymbol{v} \cdot \boldsymbol{w} d\boldsymbol{x} \\
&\quad + \frac{\beta}{\rho(m+1)} \lim_{h \to 0} \int_\Omega \frac{(|\boldsymbol{v}+h\boldsymbol{w}|^{m+1} - |\boldsymbol{v}|^{m+1})}{h} d\boldsymbol{x} \ .
\end{aligned}$$

Thanks to the Lebesgue convergence theorem [53], we get

$$
\begin{aligned}
F'(\boldsymbol{v})\cdot\boldsymbol{w} &= \frac{\mu}{\rho}\int_{\Omega} K^{-1}\boldsymbol{v}\cdot\boldsymbol{w}d\boldsymbol{x} + \frac{\beta}{\rho(m+1)}\int_{\Omega}\lim_{h\to 0}\frac{(|\boldsymbol{v}+h\boldsymbol{w}|^{m+1}-|\boldsymbol{v}|^{m+1})}{h}d\boldsymbol{x},\\
&= \frac{\mu}{\rho}\int_{\Omega} K^{-1}\boldsymbol{v}\cdot\boldsymbol{w}d\boldsymbol{x} + \frac{\beta}{\rho(m+1)}\int_{\Omega}\frac{d}{dh}|\boldsymbol{v}+h\boldsymbol{w}|^{m+1}\Big|_{h=0},\\
&= \frac{\mu}{\rho}\int_{\Omega} K^{-1}\boldsymbol{v}\cdot\boldsymbol{w}d\boldsymbol{x} + \frac{\beta}{\rho(m+1)}\int_{\Omega}\frac{d}{dh}\left[(\boldsymbol{v}+h\boldsymbol{w})\cdot(\boldsymbol{v}+h\boldsymbol{w})\right]^{\frac{m+1}{2}}\Big|_{h=0},\\
F'(\boldsymbol{v})\cdot\boldsymbol{w} &= \frac{\mu}{\rho}\int_{\Omega} K^{-1}\boldsymbol{v}\cdot\boldsymbol{w}d\boldsymbol{x} + \frac{\beta}{\rho(m+1)}\int_{\Omega}\frac{m+1}{2}|\boldsymbol{v}|^{m-1}(\boldsymbol{v}\cdot\boldsymbol{w}+\boldsymbol{v}\cdot\boldsymbol{w})d\boldsymbol{x}.
\end{aligned}
$$

Hence,

$$
F'(\boldsymbol{v})\cdot\boldsymbol{w} = \frac{\mu}{\rho}\int_{\Omega} K^{-1}\boldsymbol{v}\cdot\boldsymbol{w}d\boldsymbol{x} + \frac{\beta}{\rho}\int_{\Omega}|\boldsymbol{v}|^{m-1}(\boldsymbol{v}\cdot\boldsymbol{w})d\boldsymbol{x}. \tag{2.34}
$$

We have

$$
\begin{aligned}
F''(\boldsymbol{v})\cdot(\boldsymbol{z},\boldsymbol{w}) &= \lim_{h\to 0}\frac{f'(\boldsymbol{v}+h\boldsymbol{z})\boldsymbol{w}-f'(\boldsymbol{v})\boldsymbol{w}}{h},\\
&= \frac{\mu}{\rho}\int_{\Omega} K^{-1}\boldsymbol{z}\cdot\boldsymbol{w}d\boldsymbol{x} + \frac{\beta}{\rho}\int_{\Omega}|\boldsymbol{v}|^{m-1}(\boldsymbol{z}\cdot\boldsymbol{w})d\boldsymbol{x}\\
&\quad + \frac{\beta}{\rho}\int_{\Omega}\lim_{h\to 0}\left[\frac{|\boldsymbol{v}+h\boldsymbol{z}|^{m-1}(\boldsymbol{v}\cdot\boldsymbol{w})-|\boldsymbol{v}|^{m-1}}{h}\right](\boldsymbol{v}\cdot\boldsymbol{w})d\boldsymbol{x},\\
&= \frac{\mu}{\rho}\int_{\Omega} K^{-1}\boldsymbol{z}\cdot\boldsymbol{w}d\boldsymbol{x} + \frac{\beta}{\rho}\int_{\Omega}|\boldsymbol{v}|^{m-1}(\boldsymbol{z}\cdot\boldsymbol{w})d\boldsymbol{x}\\
&\quad + \int_{\Omega}\frac{d}{dh}|\boldsymbol{v}+h\boldsymbol{z}|^{m-1}\Big|_{h=0}(\boldsymbol{v}\cdot\boldsymbol{w})d\boldsymbol{x},\\
&= \frac{\mu}{\rho}\int_{\Omega} K^{-1}\boldsymbol{z}\cdot\boldsymbol{w}d\boldsymbol{x} + \frac{\beta}{\rho}\int_{\Omega}|\boldsymbol{v}|^{m-1}(\boldsymbol{z}\cdot\boldsymbol{w})d\boldsymbol{x}\\
&\quad + \int_{\Omega}\frac{d}{dh}\left[(\boldsymbol{v}+h\boldsymbol{z})\cdot(\boldsymbol{v}+h\boldsymbol{z})\right]^{\frac{m-1}{2}}\Big|_{h=0}(\boldsymbol{v}\cdot\boldsymbol{w})d\boldsymbol{x}.
\end{aligned}
$$

Consequently,

$$F''(\boldsymbol{v}) \cdot (\boldsymbol{z}, \boldsymbol{w}) = \frac{\mu}{\rho} \int_{\Omega} K^{-1} \boldsymbol{z} \cdot \boldsymbol{w} d\boldsymbol{x} + \frac{\beta}{\rho} \int_{\Omega} |\boldsymbol{v}|^{m-1} (\boldsymbol{z} \cdot \boldsymbol{w}) d\boldsymbol{x}$$
$$+ \frac{\beta}{\rho} \int_{\Omega} \frac{m-1}{2} |\boldsymbol{v} + h\boldsymbol{z}|^{m-3} [(\boldsymbol{v} + h\boldsymbol{z}) \cdot \boldsymbol{z} + (\boldsymbol{v} + h\boldsymbol{z}) \cdot \boldsymbol{z}] \big|_{h=0} (\boldsymbol{v} \cdot \boldsymbol{w}) d\boldsymbol{x}.$$

It follows that

$$F''(\boldsymbol{v}) \cdot (\boldsymbol{z}, \boldsymbol{w}) = \frac{\mu}{\rho} \int_{\Omega} K^{-1} \boldsymbol{z} \cdot \boldsymbol{w} d\boldsymbol{x} + \frac{\beta}{\rho} \int_{\Omega} |\boldsymbol{v}|^{m-1} (\boldsymbol{z} \cdot \boldsymbol{w}) d\boldsymbol{x}$$
$$+ \frac{\beta}{\rho} (m-1) \int_{\Omega} |\boldsymbol{v}|^{m-3} (\boldsymbol{v} \cdot \boldsymbol{z}) (\boldsymbol{v} \cdot \boldsymbol{w}) d\boldsymbol{x}. \tag{2.35}$$

Therefore,

$$F''(0) \cdot (\boldsymbol{w}, \boldsymbol{z}) = \frac{\mu}{\rho} \int_{\Omega} K^{-1} \boldsymbol{z} \cdot \boldsymbol{w} d\boldsymbol{x}, \forall \boldsymbol{w}, \forall \boldsymbol{z} \in [L^{m+1}(\Omega)]^d. \tag{2.36}$$

Now observe that F'' is positive definite and symmetric. Indeed,

$$F''(\boldsymbol{v}) \cdot (\boldsymbol{w}, \boldsymbol{w}) \geq \frac{\mu}{\rho} \lambda_s \|\boldsymbol{w}\|_{L^2}^2, \forall \boldsymbol{v}, \boldsymbol{w} \in [L^{m+1}(\Omega)]^d. \tag{2.37}$$

Let $\boldsymbol{u}$, $\boldsymbol{v}$ be in $[L^{m+1}(\Omega)]^d$. Split $\hat{\boldsymbol{u}} = \boldsymbol{u} + \boldsymbol{u}_\ell$ and $\hat{\boldsymbol{v}} = \boldsymbol{v} + \boldsymbol{u}_\ell$, where $\boldsymbol{u}_\ell$ is fixed in $[L^{m+1}(\Omega)]^d$. Then,

$$
\begin{aligned}
(F'(\hat{\boldsymbol{u}}) - F'(\hat{\boldsymbol{v}})) \cdot (\hat{\boldsymbol{u}} - \hat{\boldsymbol{v}}) &= F'(\hat{\boldsymbol{u}}) \cdot (\hat{\boldsymbol{u}} - \hat{\boldsymbol{v}}) - F'(\hat{\boldsymbol{v}}) \cdot (\hat{\boldsymbol{u}} - \hat{\boldsymbol{v}}) \\
&= \frac{\mu}{\rho} \int_\Omega (K^{-1}\hat{\boldsymbol{u}}) \cdot (\hat{\boldsymbol{u}} - \hat{\boldsymbol{v}}) d\boldsymbol{x} + \frac{\beta}{\rho} \int_\Omega |\hat{\boldsymbol{u}}|^{m-1} \left(\hat{\boldsymbol{u}} \cdot (\hat{\boldsymbol{u}} - \hat{\boldsymbol{v}})\right) d\boldsymbol{x} \\
&\quad - \left[\frac{\mu}{\rho} \int_\Omega (K^{-1}\hat{\boldsymbol{v}}) \cdot (\hat{\boldsymbol{u}} - \hat{\boldsymbol{v}}) d\boldsymbol{x} + \frac{\beta}{\rho} \int_\Omega |\hat{\boldsymbol{v}}|^{m-1} \left(\hat{\boldsymbol{v}} \cdot (\hat{\boldsymbol{u}} - \hat{\boldsymbol{v}})\right) d\boldsymbol{x} \right] \\
&= \frac{\mu}{\rho} \int_\Omega (K^{-1}(\hat{\boldsymbol{u}} - \hat{\boldsymbol{v}}) \cdot (\hat{\boldsymbol{u}} - \hat{\boldsymbol{v}}) d\boldsymbol{x} \\
&\quad + \frac{\beta}{\rho} \int_\Omega \left(|\hat{\boldsymbol{u}}|^{m-1}\hat{\boldsymbol{u}} - |\hat{\boldsymbol{v}}|^{m-1}\hat{\boldsymbol{v}} \right) \cdot (\hat{\boldsymbol{u}} - \hat{\boldsymbol{v}}) d\boldsymbol{x}.
\end{aligned}
\tag{2.38}
$$

That is,

$$
(F'(\hat{\boldsymbol{u}}) - F'(\hat{\boldsymbol{v}})) \cdot (\hat{\boldsymbol{u}} - \hat{\boldsymbol{v}}) = \int_\Omega (\mathcal{A}(\hat{\boldsymbol{u}}) - \mathcal{A}(\hat{\boldsymbol{v}})) \cdot (\boldsymbol{u} - \boldsymbol{v}) d\boldsymbol{x} \tag{2.39}
$$

Furthermore, by the mean value theorem for integrals see [8, Theorem 1.43], we obtain

$$
(F'(\hat{\boldsymbol{u}}) - F'(\hat{\boldsymbol{v}})) \cdot (\hat{\boldsymbol{u}} - \hat{\boldsymbol{v}}) = \int_0^1 F''(\hat{\boldsymbol{v}} + \alpha(\hat{\boldsymbol{u}} - \hat{\boldsymbol{v}})) \cdot (\hat{\boldsymbol{u}} - \hat{\boldsymbol{v}}, \hat{\boldsymbol{u}} - \hat{\boldsymbol{v}}) d\alpha.
$$

Therefore, (2.38) becomes,

$$
\int_\Omega (\mathcal{A}(\hat{\boldsymbol{u}}) - \mathcal{A}(\hat{\boldsymbol{v}})) \cdot (\boldsymbol{u} - \boldsymbol{v}) d\boldsymbol{x} = \int_0^1 F''(\hat{\boldsymbol{v}} + \alpha(\hat{\boldsymbol{u}} - \hat{\boldsymbol{v}})) \cdot (\hat{\boldsymbol{u}} - \hat{\boldsymbol{v}}, \hat{\boldsymbol{u}} - \hat{\boldsymbol{v}}) d\alpha, \tag{2.40}
$$

Thanks to (2.37), equation (2.40) becomes,

$$\begin{aligned}\int_{\Omega}(\mathcal{A}(\hat{\boldsymbol{u}})-\mathcal{A}(\hat{\boldsymbol{v}}))\cdot(\boldsymbol{u}-\boldsymbol{v})d\boldsymbol{x} &\geq \int_0^1 \frac{\mu}{\rho}\lambda_s\|\hat{\boldsymbol{u}}-\hat{\boldsymbol{v}}\|_{L^2}^2 d\alpha \\ &\geq \frac{\mu}{\rho}\lambda_s\|\hat{\boldsymbol{u}}-\hat{\boldsymbol{v}}\|_{[L^m(\Omega)]^d}^2 \end{aligned} \tag{2.41}$$

Hence, we have (2.33), since $\hat{\boldsymbol{u}}-\hat{\boldsymbol{v}}=\boldsymbol{u}-\boldsymbol{v}$. ∎

Lemma 2.5 *The map $\boldsymbol{u}\mapsto\mathcal{A}(\boldsymbol{u}+\boldsymbol{u}_\ell)$ in* (2.6) *is coercive in $[L^{m+1}(\Omega)]^d$, for any $\boldsymbol{u}_\ell$ fixed in $[L^{m+1}(\Omega)]^d$,*

$$\lim_{\|\boldsymbol{u}\|_{[L^{m+1}(\Omega)]^d}\to\infty}\left(\frac{1}{\|\boldsymbol{u}\|_{[L^{m+1}(\Omega)]^d}}\int_{\Omega}\mathcal{A}(\boldsymbol{u}+\boldsymbol{u}_{\boldsymbol{\ell}})\cdot\boldsymbol{u}d\boldsymbol{x}\right)=\infty. \tag{2.42}$$

Proof. Let $\boldsymbol{u}\in[L^{m+1}(\Omega)]^d$ be arbitrary chosen and assign $\hat{\boldsymbol{u}}=\boldsymbol{u}+\boldsymbol{u}_\ell$. Then,

$$\int_{\Omega}\mathcal{A}(\hat{\boldsymbol{u}})\cdot\boldsymbol{u}d\boldsymbol{x}=\int_{\Omega}\mathcal{A}(\hat{\boldsymbol{u}})\hat{\boldsymbol{u}}d\boldsymbol{x}-\int_{\Omega}\mathcal{A}(\hat{\boldsymbol{u}})\boldsymbol{u}_\ell d\boldsymbol{x} \tag{2.43}$$

Thanks to (2.34), we see that,

$$\begin{aligned}F'(\hat{\boldsymbol{u}})\cdot\hat{\boldsymbol{u}} &= \frac{\mu}{\rho}\int_{\Omega}K^{-1}\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{u}}d\boldsymbol{x}+\frac{\beta}{\rho}\int_{\Omega}|\hat{\boldsymbol{u}}|^{m-1}(\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{u}})d\boldsymbol{x} \\ &= \int_{\Omega}\left(\frac{\mu}{\rho}K^{-1}\hat{\boldsymbol{u}}+\frac{\beta}{\rho}|\boldsymbol{u}|^{m-1}\hat{\boldsymbol{u}}\right)\cdot\hat{\boldsymbol{u}}d\boldsymbol{x} \\ &= \int_{\Omega}\mathcal{A}(\hat{\boldsymbol{u}})\cdot\hat{\boldsymbol{u}}d\boldsymbol{x}.\end{aligned}$$

With the last line above, (2.43) becomes,

$$\int_\Omega \mathcal{A}(\hat{\boldsymbol{u}}) \cdot \boldsymbol{u} d\boldsymbol{x} = F'(\hat{\boldsymbol{u}}) \cdot \hat{\boldsymbol{u}} - \int_\Omega \mathcal{A}(\hat{\boldsymbol{u}}) \boldsymbol{u}_\ell d\boldsymbol{x} \tag{2.44}$$

$$\begin{aligned}
F'(\hat{\boldsymbol{u}}) \cdot \hat{\boldsymbol{u}} &= \frac{\mu}{\rho} \int_\Omega (K^{-1}\hat{\boldsymbol{u}}) \cdot \hat{\boldsymbol{u}} d\boldsymbol{x} + \frac{\beta}{\rho} \int_\Omega |\hat{\boldsymbol{u}}|^{m-1} (\hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{u}}) d\boldsymbol{x} \\
&= \frac{\mu}{\rho} \int_\Omega (K^{-1}\hat{\boldsymbol{u}}) \cdot \hat{\boldsymbol{u}} d\boldsymbol{x} + \frac{\beta}{\rho} \int_\Omega |\hat{\boldsymbol{u}}|^{m+1} d\boldsymbol{x} \\
F'(\hat{\boldsymbol{u}}) \cdot \hat{\boldsymbol{u}} &\geq \frac{\mu}{\rho} \lambda_s \|\hat{\boldsymbol{u}}\|^2_{L^2} + \frac{\beta}{\rho} \|\hat{\boldsymbol{u}}\|^{m+1}_{[L^{m+1}(\Omega)]^d} \\
F'(\hat{\boldsymbol{u}}) \cdot \hat{\boldsymbol{u}} &\geq \frac{C\mu}{\rho} \lambda_s \|\hat{\boldsymbol{u}}\|^2_{[L^m(\Omega)]^d} + \frac{\beta}{\rho} \|\hat{\boldsymbol{u}}\|^{m+1}_{[L^{m+1}(\Omega)]^d}, m \leq 2. \qquad (2.45)
\end{aligned}$$

Now, we estimate the second term in the RHS of (2.44).

$$\begin{aligned}
\int_\Omega \mathcal{A}(\hat{\boldsymbol{u}}) \boldsymbol{u}_\ell d\boldsymbol{x} &\leq \|\mathcal{A}(\hat{\boldsymbol{u}})\|_{[L^{\frac{m+1}{m}}(\Omega)]^d} \|\boldsymbol{u}_\ell\|_{[L^{m+1}(\Omega)]^d} \\
&\leq \left(\frac{\mu}{\rho} \|K^{-1}\|_{L^\infty(\Omega)} \|\hat{\boldsymbol{u}}\|_{[L^{\frac{m+1}{m}}(\Omega)]^d} + \frac{\beta}{\rho} \|\hat{\boldsymbol{u}}\|^m_{[L^{m+1}(\Omega)]^d} \right) \|\boldsymbol{u}_\ell\|_{[L^{m+1}(\Omega)]^d} \\
\int_\Omega \mathcal{A}(\hat{\boldsymbol{u}}) \boldsymbol{u}_\ell d\boldsymbol{x} &\leq \frac{\mu}{\rho} \|K^{-1}\|_{L^\infty(\Omega)} \|\hat{\boldsymbol{u}}\|_{[L^{\frac{m+1}{m}}(\Omega)]^d} \|\boldsymbol{u}_\ell\|_{[L^{m+1}(\Omega)]^d} \\
&\quad + \frac{\beta}{\rho} \|\hat{\boldsymbol{u}}\|^m_{[L^{m+1}(\Omega)]^d} \|\boldsymbol{u}_\ell\|_{[L^{m+1}(\Omega)]^d} \qquad (2.46)
\end{aligned}$$

Substituting (2.45) and (2.46) in (2.44), we find

$$\begin{aligned}
\int_\Omega \mathcal{A}(\hat{\boldsymbol{u}}) \cdot \boldsymbol{u} d\boldsymbol{x} \geq &\frac{C\mu}{\rho} \lambda_s \|\hat{\boldsymbol{u}}\|^2_{[L^m(\Omega)]^d} + \frac{\beta}{\rho} \|\hat{\boldsymbol{u}}\|^{m+1}_{[L^{m+1}(\Omega)]^d} \\
&- \frac{\mu}{\rho} \|K^{-1}\|_{L^\infty(\Omega)} \|\hat{\boldsymbol{u}}\|_{[L^{\frac{m+1}{m}}(\Omega)]^d} \|\boldsymbol{u}_\ell\|_{[L^{m+1}(\Omega)]^d} \\
&- \frac{\beta}{\rho} \|\hat{\boldsymbol{u}}\|^m_{[L^{m+1}(\Omega)]^d} \|\boldsymbol{u}_\ell\|_{[L^{m+1}(\Omega)]^d}, \qquad (2.47)
\end{aligned}$$

Hence,

$$\int_\Omega \mathcal{A}(\hat{\boldsymbol{u}}) \cdot \boldsymbol{u} d\boldsymbol{x} \geq \frac{\beta}{\rho} \|\hat{\boldsymbol{u}}\|^m_{[L^{m+1}(\Omega)]^d} \left(\|\hat{\boldsymbol{u}}\|_{[L^{m+1}(\Omega)]^d} - \|\boldsymbol{u}_\ell\|_{[L^{m+1}(\Omega)]^d} \right) + \frac{\mu}{\rho} C\lambda_s \|\hat{\boldsymbol{u}}\|_{[L^m(\Omega)]^d} \left(\|\hat{\boldsymbol{u}}\|_{[L^m(\Omega)]^d} - \frac{\|K^{-1}\|_{L^\infty(\Omega)} \|\boldsymbol{u}_\ell\|_{[L^{m+1}(\Omega)]^d}}{C\lambda_s} \right). \tag{2.48}$$

If $\|\boldsymbol{u}\|_{[L^m(\Omega)]^d} \longrightarrow \infty$, then

$$\frac{\mu}{\rho} C\lambda_s \|\boldsymbol{u}\|_{[L^m(\Omega)]^d} \left(\|\hat{\boldsymbol{u}}\|_{[L^m(\Omega)]^d} - \frac{\|K^{-1}\|_{L^\infty(\Omega)} \|\boldsymbol{u}_\ell\|_{[L^{m+1}(\Omega)]^d}}{C\lambda_s} \right) \geq 0.$$

Therefore,

$$\int_\Omega \mathcal{A}(\hat{\boldsymbol{u}}) \boldsymbol{u} d\boldsymbol{x} \geq \frac{\beta}{\rho} \|\hat{\boldsymbol{u}}\|_{[L^{m+1}(\Omega)]^d} \|\hat{\boldsymbol{u}}\|^{m-1}_{[L^{m+1}(\Omega)]^d} \left(\|\hat{\boldsymbol{u}}\|_{[L^{m+1}(\Omega)]^d} - \|\boldsymbol{u}_\ell\|_{[L^{m+1}(\Omega)]^d} \right).$$

Recall,

$$\hat{\boldsymbol{u}} = \boldsymbol{u} + \boldsymbol{u}_\ell \implies \|\hat{\boldsymbol{u}}\|_{[L^{m+1}(\Omega)]^d} \geq \|\boldsymbol{u}\|_{[L^{m+1}(\Omega)]^d} - \|\boldsymbol{u}_\ell\|_{[L^{m+1}(\Omega)]^d} .$$

As $\|\boldsymbol{u}\|_{[L^m(\Omega)]^d} \longrightarrow \infty$,

$$\|\boldsymbol{u}\|_{[L^{m+1}(\Omega)]^d} - \|\boldsymbol{u}_\ell\|_{[L^{m+1}(\Omega)]^d} \geq \frac{1}{2} \|\boldsymbol{u}\|_{[L^{m+1}(\Omega)]^d},$$

since $\|\boldsymbol{u}\|_{[L^{m+1}(\Omega)]^d} \geq 2\|\boldsymbol{u}_\ell\|_{[L^{m+1}(\Omega)]^d}$.

Hence,

$$\int_\Omega \mathcal{A}(\hat{\boldsymbol{u}})\boldsymbol{u}d\boldsymbol{x} \geq \frac{\beta}{2\rho}\|\boldsymbol{u}\|_{[L^{m+1}(\Omega)]^d}\|\hat{\boldsymbol{u}}\|^{m-1}_{[L^{m+1}(\Omega)]^d}\left(\|\hat{\boldsymbol{u}}\|_{[L^{m+1}(\Omega)]^d} - \|\boldsymbol{u}_\ell\|_{[L^{m+1}(\Omega)]^d}\right),$$

It follows that,

$$\frac{1}{\|\boldsymbol{u}\|_{[L^{m+1}(\Omega)]^d}}\int_\Omega \mathcal{A}(\hat{\boldsymbol{u}})\boldsymbol{u}d\boldsymbol{x} \geq \|\hat{\boldsymbol{u}}\|^{m-1}_{[L^{m+1}(\Omega)]^d}\left(\|\hat{\boldsymbol{u}}\|_{[L^{m+1}(\Omega)]^d} - \|\boldsymbol{u}_\ell\|_{[L^{m+1}(\Omega)]^d}\right).$$

Letting $\|\boldsymbol{u}\|_{[L^{m+1}(\Omega)]^d} \longrightarrow \infty$, the desired result (2.42) is achieved. ∎

Proposition 2.3 *For $\boldsymbol{u}_\ell$ fixed in $[L^{m+1}(\Omega)]^d$, the map,*

$$\theta \mapsto \int_\Omega \mathcal{A}(\boldsymbol{u} + \boldsymbol{u}_\ell + \theta\boldsymbol{v})\cdot\boldsymbol{w}d\boldsymbol{x},$$

is continuous on $\mathbb{R}$, $\forall\boldsymbol{u} \in [L^{m+1}(\Omega)]^d$ and $\forall\boldsymbol{v} \in [L^{m+1}(\Omega)]^d$. In other words, $\mathcal{A}$ is hemi-continuous in $[L^{m+1}(\Omega)]^d$.

Proof. Let $\boldsymbol{u}$ and $\boldsymbol{v}$ in $[L^{m+1}(\Omega)]^d$ be arbitrary functions and assign $\hat{\boldsymbol{u}} = \boldsymbol{u}+\boldsymbol{u}_\ell$. For any $\theta_1, \theta_2 \in \mathbb{R}$,

$$\begin{aligned}\int_\Omega \mathcal{A}(\hat{\boldsymbol{u}} + \theta_1\boldsymbol{v})\cdot\boldsymbol{v}d\boldsymbol{x} &= \int_\Omega\left(\frac{\mu}{\rho}K^{-1}(\hat{\boldsymbol{u}}+\theta_1\boldsymbol{v}) + \frac{\beta}{\rho}|\hat{\boldsymbol{u}}+\theta_1\boldsymbol{v}|^{m-1}\cdot(\hat{\boldsymbol{u}}+\theta_1\boldsymbol{v})\right)\cdot\boldsymbol{v}d\boldsymbol{x}.\\ &= \frac{\mu}{\rho}\int_\Omega K^{-1}(\hat{\boldsymbol{u}}+\theta_1\boldsymbol{v})\cdot\boldsymbol{v}d\boldsymbol{x} + \frac{\beta}{\rho}\int_\Omega|\hat{\boldsymbol{u}}+\theta_1\boldsymbol{v}|^{m-1}(\hat{\boldsymbol{u}}+\theta_1\boldsymbol{v})\cdot\boldsymbol{v}d\boldsymbol{x}.\end{aligned}$$

Thanks to (2.34), the last line becomes

$$\int_\Omega \mathcal{A}(\hat{\boldsymbol{u}}+\theta_1\boldsymbol{v})\cdot\boldsymbol{v}d\boldsymbol{x}=F'(\hat{\boldsymbol{u}}+\theta_1\boldsymbol{v})\cdot\boldsymbol{v}.$$

Similarly,

$$\int_\Omega \mathcal{A}(\hat{\boldsymbol{u}}+\theta_2\boldsymbol{v})\cdot\boldsymbol{v}d\boldsymbol{x}=F'(\hat{\boldsymbol{u}}+\theta_2\boldsymbol{v})\cdot\boldsymbol{v}.$$

Therefore,

$$\int_\Omega (\mathcal{A}(\hat{\boldsymbol{u}}+\theta_1\boldsymbol{v})-\mathcal{A}(\hat{\boldsymbol{u}}+\theta_2\boldsymbol{v}))\cdot\boldsymbol{v}d\boldsymbol{x}=(F'(\hat{\boldsymbol{u}}+\theta_1\boldsymbol{v})-F'(\hat{\boldsymbol{u}}+\theta_2\boldsymbol{v}))\cdot\boldsymbol{v}. \quad (2.49)$$

Now, we define

$$H(\theta)=F'\left(\theta(\hat{\boldsymbol{u}}+\theta_1\boldsymbol{v})+(1-\theta)(\hat{\boldsymbol{u}}+\theta_2\boldsymbol{v})\right),\quad \theta\in[0,1].$$

By the mean value theorem [8],

$$H(1)-H(0)=H'(\alpha),\quad \alpha\in(0,1). \quad (2.50)$$

Infact,

$$\begin{aligned} H'(\alpha)&=F''\left(\alpha(\hat{\boldsymbol{u}}+\theta_1\boldsymbol{v})+(1-\alpha)(\hat{\boldsymbol{u}}+\theta_2\boldsymbol{v})\right)\cdot(\theta_1-\theta_2)\boldsymbol{v},\\ &=F''\left(\hat{\boldsymbol{u}}+\theta_2\boldsymbol{v}-\alpha(\theta_2-\theta_1)\boldsymbol{v}\right)\cdot(\theta_1-\theta_2)\boldsymbol{v}. \end{aligned} \quad (2.51)$$

Therefore, (2.50) becomes

$$(F'(\hat{\boldsymbol{u}}+\theta_1\boldsymbol{v})-F'(\hat{\boldsymbol{u}}+\theta_2\boldsymbol{v}))\cdot\boldsymbol{v}=F''\left(\hat{\boldsymbol{u}}+\theta_2\boldsymbol{v}-\alpha(\theta_2-\theta_1)\boldsymbol{v}\right)\cdot(\theta_1-\theta_2)\boldsymbol{v},\ \alpha\in(0,1). \tag{2.52}$$

Using (2.52), It follows from (2.49), that

$$\int_\Omega(\mathcal{A}(\hat{\boldsymbol{u}}+\theta_1\boldsymbol{v})-\mathcal{A}(\hat{\boldsymbol{u}}+\theta_2\boldsymbol{v}))\cdot\boldsymbol{v}d\boldsymbol{x}=-(\theta_2-\theta_1)\int_0^1F''\left(\hat{\boldsymbol{u}}+\theta_2\boldsymbol{v}-\alpha(\theta_2-\theta_1)\boldsymbol{v}\right)\cdot(\boldsymbol{v},\boldsymbol{v})d\alpha \tag{2.53}$$

In view of (2.35), the RHS term in (2.53) goes to zero as $\theta_2-\theta_1$ goes to zero. Thus Proposition (2.3) is proved. ∎

Theorem 2.1 *Provided* $b\in L^{\frac{d(m+1)}{d+(m+1)}}(\Omega)$ *and* $g\in L^{\frac{(d-1)(m+1)}{d}}(\Gamma)$ *satisfying* (1.4), *Problem* (2.1)-(2.2) *has a unique solution* $(\boldsymbol{u},p)$, *with* $\boldsymbol{u}\in[L^{m+1}(\Omega)]^d$ *and* $p\in W^{1,\frac{m+1}{m}}(\Omega)\cap L^2_0(\Omega)$.

Proof.

In view of Lemmas 2.3, 2.4, 2.5 and Proposition 2.3, the existence and uniqueness of $\boldsymbol{u}$ follows from Theorem 1.1. The inf-sup condition in Lemma 2.1 guarantees the existence and uniqueness of p so that the pair $(\boldsymbol{u},p)$ solves Problem (2.1)-(2.2). ∎

2.3 Stability

Theorem 2.2 *For any lifting* $\boldsymbol{u}_\ell$ *in* $[L^{m+1}(\Omega)]^d$ *satisfying* (4.30)*, the pair of function* $(\boldsymbol{u}, p)$ *in Theorem 2.1 satisfies the following estimates:*

$$\|\boldsymbol{u}\|_{[L^{m+1}(\Omega)]^d} \le \Bigg(\|\boldsymbol{u_\ell}\|^{m+1}_{[L^{m+1}(\Omega)]^d} + \frac{C(m+1)}{4}\frac{\mu}{\beta}\frac{\|K^{-1}\|_{L^\infty(\Omega)}}{\lambda_s}\|\boldsymbol{u_\ell}\|^2_{[L^{m+1}(\Omega)]^d} + \frac{\rho(m+1)}{\beta}\|\boldsymbol{f}\|_{[L^{\frac{m+1}{m}}(\Omega)]^d}\|\boldsymbol{u}_0\|_{[L^{m+1}(\Omega)]^d}\Bigg)^{\frac{1}{m+1}}, \tag{2.54}$$

and

$$\|\nabla p\|_{[L^{\frac{m+1}{m}}(\Omega)]^d} \le \frac{\mu}{\rho}\|K^{-}\|_{L^\infty}\|\boldsymbol{u}\|_{[L^{\frac{m+1}{m}}(\Omega)]^d} + \frac{\beta}{\rho}\|\boldsymbol{u}\|^2_{[L^{m+1}(\Omega)]^d} + \|\boldsymbol{f}\|_{[L^{\frac{m+1}{m}}(\Omega)]^d}\ . \tag{2.55}$$

Proof. Setting $\boldsymbol{\varphi} = \boldsymbol{u}_0$ in (2.22) gives

$$\int_\Omega \mathcal{A}(\boldsymbol{u}_0 + \boldsymbol{u}_\ell)\cdot\boldsymbol{u}_0 d\boldsymbol{x} = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}_0 d\boldsymbol{x}. \tag{2.56}$$

Therefore,

$$\begin{aligned}\int_\Omega \mathcal{A}(\boldsymbol{u})\cdot\boldsymbol{u} d\boldsymbol{x} &= \int_\Omega \mathcal{A}(\boldsymbol{u}_0 + \boldsymbol{u}_\ell)\cdot\boldsymbol{u}_0 d\boldsymbol{x} + \int_\Omega \mathcal{A}(\boldsymbol{u}_0 + \boldsymbol{u}_\ell)\cdot\boldsymbol{u}_\ell d\boldsymbol{x},\\ &= \int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}_0 d\boldsymbol{x} + \int_\Omega \mathcal{A}(\boldsymbol{u}_0 + \boldsymbol{u}_\ell)\cdot\boldsymbol{u}_\ell d\boldsymbol{x}.\end{aligned} \tag{2.57}$$

We estimate the LHS of (2.57) as follows:

$$\begin{aligned}\int_\Omega \mathcal{A}(\boldsymbol{u})\cdot\boldsymbol{u}d\boldsymbol{x} &= \int_\Omega \frac{\mu}{\rho}K^{-1}\boldsymbol{u}\cdot\boldsymbol{u}d\boldsymbol{x} + \int_\Omega \frac{\beta}{\rho}|\boldsymbol{u}|^{m-1}\boldsymbol{u}\cdot\boldsymbol{u}d\boldsymbol{x},\\ &\geq \frac{\mu\lambda_s}{\rho}\|\boldsymbol{u}\|^2_{[L^2(\Omega)]^d} + \frac{\beta}{\rho}\|\boldsymbol{u}\|^{m+1}_{[L^{m+1}(\Omega)]^d}.\end{aligned} \tag{2.58}$$

The RHS of (2.57) gives the following bounds:

$$\begin{aligned}\int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}_0 d\boldsymbol{x} + \int_\Omega \mathcal{A}(\boldsymbol{u})\cdot\boldsymbol{u}_\ell d\boldsymbol{x} &\leq \|\boldsymbol{f}\|_{[L^{\frac{m+1}{m}}(\Omega)]^d}\|\boldsymbol{u}_0\|_{[L^{m+1}(\Omega)]^d}\\ &+ \|\mathcal{A}(\boldsymbol{u})\|_{[L^{\frac{m+1}{m}}(\Omega)]^d}\|\boldsymbol{u}_\ell\|_{[L^{m+1}(\Omega)]^d}.\end{aligned}$$

In view of (2.30), we get

$$\begin{aligned}\int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}_0 d\boldsymbol{x} + \int_\Omega \mathcal{A}(\boldsymbol{u})\cdot\boldsymbol{u}_\ell d\boldsymbol{x} &\leq \|\boldsymbol{f}\|_{[L^{\frac{m+1}{m}}(\Omega)]^d}\|\boldsymbol{u}_0\|_{[L^{m+1}(\Omega)]^d}\\ &+ \frac{\mu}{\rho}\|K^{-1}\|_{L^\infty(\Omega)}\|\boldsymbol{u}\|_{[L^{\frac{m+1}{m}}(\Omega)]^d}\|\boldsymbol{u}_\ell\|_{[L^{m+1}(\Omega)]^d}\\ &+ \frac{\beta}{\rho}\|\boldsymbol{u}\|^m_{[L^{m+1}(\Omega)]^d}\|\boldsymbol{u}_\ell\|_{[L^{m+1}(\Omega)]^d}.\end{aligned} \tag{2.59}$$

Substituting (2.58) and (2.59) into (2.57) yields

$$\begin{aligned}\frac{\mu}{\rho}\lambda_s\|\boldsymbol{u}\|^2_{L^2} + \frac{\beta}{\rho}\|\boldsymbol{u}\|^{m+1}_{[L^{m+1}(\Omega)]^d} &\leq \frac{\mu}{\rho}\|K^{-1}\|_{L^\infty(\Omega)}\|\boldsymbol{u}\|_{[L^{\frac{m+1}{m}}(\Omega)]^d}\|\boldsymbol{u}_\ell\|_{[L^{m+1}(\Omega)]^d}\\ &+ \frac{\beta}{\rho}\|\boldsymbol{u}\|^m_{[L^{m+1}(\Omega)]^d}\|\boldsymbol{u}_\ell\|_{[L^{m+1}(\Omega)]^d}\\ &+ \|\boldsymbol{f}\|_{[L^{\frac{m+1}{m}}(\Omega)]^d}\|\boldsymbol{u}_0\|_{[L^{m+1}(\Omega)]^d}\end{aligned} \tag{2.60}$$

Then for any $\epsilon > 0$ and $m > 1$,

$$\frac{\mu}{\rho}\lambda_s\|\boldsymbol{u}\|_{L^2}^2 + \frac{\beta}{\rho}\|\boldsymbol{u}\|_{[L^{m+1}(\Omega)]^d}^{m+1} \leq \frac{\mu}{2\rho}\|K^{-1}\|_{L^\infty(\Omega)}\left(\epsilon\|\boldsymbol{u}\|_{[L^{\frac{m+1}{m}}(\Omega)]^d}^2 + \frac{1}{\epsilon}\|\boldsymbol{u}_\ell\|_{[L^{m+1}(\Omega)]^d}^2\right)$$
$$+ \frac{\beta}{\rho(m+1)}\left(m\|\boldsymbol{u}\|_{[L^{m+1}(\Omega)]^d}^{m+1} + \|\boldsymbol{u}_\ell\|_{[L^{m+1}(\Omega)]^d}^{m+1}\right)$$
$$+ \|\boldsymbol{f}\|_{[L^{\frac{m+1}{m}}(\Omega)]^d}\|\boldsymbol{u}_0\|_{[L^{m+1}(\Omega)]^d}. \tag{2.61}$$

Since $2 > \frac{m+1}{m}$, for any $m \in (1, 2]$, we get

$$\frac{\mu}{\rho}\lambda_s\|\boldsymbol{u}\|_{[L^2(\Omega)]^d}^2 + \frac{\beta}{\rho}\|\boldsymbol{u}\|_{[L^{m+1}(\Omega)]^d}^{m+1} \leq \frac{\mu}{2\rho}\|K^{-1}\|_{L^\infty(\Omega)}\left(C\epsilon\|\boldsymbol{u}\|_{[L^2(\Omega)]^d]}^2 + \frac{1}{\epsilon}\|\boldsymbol{u}_\ell\|_{[L^{m+1}(\Omega)]^d}^2\right)$$
$$+ \frac{\beta}{\rho(m+1)}\left(m\|\boldsymbol{u}\|_{[L^{m+1}(\Omega)]^d}^{m+1} + \|\boldsymbol{u}_\ell\|_{[L^{m+1}(\Omega)]^d}^{m+1}\right)$$
$$+ \|\boldsymbol{f}\|_{[L^{\frac{m+1}{m}}(\Omega)]^d}\|\boldsymbol{u}_0\|_{[L^{m+1}(\Omega)]^d}.$$

choose $\epsilon = \frac{2\lambda_s}{C\|K^{-1}\|_{L^\infty(\Omega)}}$.

The last line inequality above becomes:

$$\frac{\beta}{\rho(m+1)}\|\boldsymbol{u}\|_{[L^{m+1}(\Omega)]^d}^{m+1} \leq \frac{\beta}{(m+1)}\|\boldsymbol{u}_\ell\|_{[L^{m+1}(\Omega)]^d}^{m+1} + \frac{C}{4}\frac{\mu}{\rho}\frac{\|K^{-1}\|_{L^\infty(\Omega)}^2}{\lambda_s}\|\boldsymbol{u}_\ell\|_{[L^{m+1}(\Omega)]^d}^2$$
$$+ \|\boldsymbol{f}\|_{[L^{\frac{m+1}{m}}(\Omega)]^d}\|\boldsymbol{u}_0\|_{[L^{m+1}(\Omega)]^d}. \tag{2.62}$$

Hence (2.54) follows.

In view of (2.23),

$$\|\nabla p\|_{[L^{\frac{m+1}{m}}(\Omega)]^d} \leq \|\boldsymbol{\mathcal{A}}(\boldsymbol{u}_0 + \boldsymbol{u}_\ell)\|_{[L^{\frac{m+1}{m}}(\Omega)]^d} + \|\boldsymbol{f}\|_{[L^{\frac{m+1}{m}}(\Omega)]^d} \tag{2.63}$$

Now applying (2.30) to the RHS of (2.63), we get

$$\|\nabla p\|_{[L^{\frac{m+1}{m}}(\Omega)]^d} \leq \frac{\mu}{\rho}\|K^{-1}\|_{L^\infty(\Omega)}\|\boldsymbol{u}_0+\boldsymbol{u}_\ell\|_{[L^{\frac{m+1}{m}}(\Omega)]^d}+\frac{\beta}{\rho}\|\boldsymbol{u}_0+\boldsymbol{u}_\ell\|^m_{[L^{m+1}(\Omega)]^d}+\|\boldsymbol{f}\|_{[L^{\frac{m+1}{m}}(\Omega)]^d}.$$

Thus (2.55) is established, since $\boldsymbol{u} = \boldsymbol{u}_0 + \boldsymbol{u}_\ell$.

$\blacksquare$

CHAPTER 3

MIXED FINITE ELEMENT METHOD FOR GENERALIZED DARCY-FORCHHEIMER EQUATION

In this Chapter, we formulate and analyze the standard mixed finite element method for the generalized Darcy-Forchheimer equation. Through out this chapter, the variable $m \in (1, 2]$, and

$$m_i = \frac{m}{m^i - 1} \quad \text{for} \quad i = 1,\, 2, \quad \text{and} \;\; \bar{m} = \frac{m(m+1)}{m^2 - m - 1} \quad .$$

3.1 Discretization

Let Ω be a polygon ($d = 2$) or a Liptschitz polyhedron ($d = 3$). We assume $\mathcal{T}_h$ is a family of conforming triangulations of $\bar{\Omega}$. For each $T \in \mathcal{T}_h$, let h_T denote the diameter of T, ρ_T denote the diameter of the sphere inscribed in T and $h = \max_{T \in \mathcal{T}_h} h_T$. We assume that our mesh is regular [17], that is

$$\forall T \in \mathcal{T}_h, \frac{h_T}{\rho_T} \leq \alpha, \quad \text{for some positive constant independent of} \quad h \text{ and } T. \tag{3.1}$$

We recall the following interpolations and their convergence properties. The piecewise constant projection $\Pi_h : [L^1(\Omega)]^d \longrightarrow X_h$ is defined by

$$\forall T \in \boldsymbol{\mathcal{T}}, \forall \boldsymbol{u} \in L^1(T), \Pi_h \boldsymbol{u}|_T = \frac{1}{|T|} \int_T \boldsymbol{u}(\boldsymbol{x}) d\boldsymbol{x}. \tag{3.2}$$

Some properties of Π_h [30, 9] are presented as follows. For any $1 \leq r < \infty$.

$$\lim_{h \mapsto 0} \Pi_h \boldsymbol{u} = \boldsymbol{u} \ \text{ strongly in } \ L^r(\Omega) \ \ \forall \boldsymbol{u} \in L^r(\Omega). \tag{3.3}$$

$$\|\boldsymbol{u} - \Pi_h \boldsymbol{u}\|_{L^r} = O(h), \quad \forall \boldsymbol{u} \in W^{1,r}(\Omega). \tag{3.4}$$

Furthermore, let I_h be a linear polynomial interpolation operator. Then the following properties holds [30, 9], for any $1 \leq q < \infty$,

$$\|(p - I_h p)\|_{L^q} + h\|\nabla(p - I_h p)\|_{L^q} \leq Ch^2 \|p\|_{W^{2,q}}, \tag{3.5}$$

Next, we introduce the finite dimensional spaces.

$$X_h := \left\{ \boldsymbol{v} \in [L^2(\Omega)]^2; \boldsymbol{v}|_T \in \mathbb{P}_0 \text{ for all } T \in \mathcal{T}_h \right\} \subset X, \tag{3.6}$$

$$Q_h := \left\{ q \in C^0(\bar{\Omega}); q|_T \in \mathbb{P}_1 \text{ for all } T \in \mathcal{T}_h \right\}, \tag{3.7}$$

$$M_h := Q_h \cap L_0^2(\Omega) \subset M, \tag{3.8}$$

where $\mathbb{P}_k$ denotes the space of polynomials of degree $\leq k$.

3.1.1 Discrete variational formulation

Let $\boldsymbol{u}_h \in X_h$ be the vector approximation of the velocity $\boldsymbol{u}$ and $p_h \in M_h$ be the scalar approximation of the pressure p, which are defined through the discrete variational formulation of (2.1)-(2.2) as follows: For any $\boldsymbol{f} \in [L^{\frac{m+1}{m}}(\Omega)]^d$,

$$\frac{\mu}{\rho}\int_\Omega (\boldsymbol{K}^{-1}\boldsymbol{u}_h)\cdot\boldsymbol{\varphi}_h d\boldsymbol{x} + \frac{\beta}{\rho}\int_\Omega |\boldsymbol{u}_h|^{m-1}\boldsymbol{u}_h\cdot\boldsymbol{\varphi}_h d\boldsymbol{x} + \int_\Omega \nabla p_h\cdot\boldsymbol{\varphi}_h d\boldsymbol{x} = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{\varphi} d\boldsymbol{x}, \tag{3.9}$$

$$\int_\Omega \nabla q_h \cdot \boldsymbol{u}_h d\boldsymbol{x} = -\int_\Omega b q_h \boldsymbol{d}x + \int_\Gamma g q_h \boldsymbol{d}\sigma, \tag{3.10}$$

$\forall q_h \in M_h, \forall \boldsymbol{\varphi}_h \in X_h$. We focus next on the existence, uniqueness and stability of the above scheme. Let V_h and $V_h^{\perp}$ be defined as follows:

$$V_h = \left\{ \boldsymbol{v}_h \in X_h; \sum_{T\in\mathcal{T}_h} \int_T \bigtriangledown q_h \cdot \boldsymbol{v}_h d\boldsymbol{x} = 0, \quad \forall q_h \in M_h \right\}, \tag{3.11}$$

$$V_h^{\perp} = \left\{ \boldsymbol{v}_h \in X_h; \int_\Omega \boldsymbol{v}_h \cdot \boldsymbol{w}_h d\boldsymbol{x} = 0, \quad \forall \boldsymbol{w}_h \in V_h \right\}. \tag{3.12}$$

The following proposition is a special case of [31, Proposition 6]

Proposition 3.1 *The discrete spaces satisfy the following infsup condition: for* $m \in (1, 2]$,

$$\sup_{\boldsymbol{\phi}_h \in X_h} \left(\frac{1}{\|\boldsymbol{\phi}_h\|_{[L^{m+1}(\Omega)]^d}} \sum_{T\in\mathcal{T}_h} \int_T \bigtriangledown q_h \cdot \boldsymbol{\phi}_h d\boldsymbol{x} \right) = \| \bigtriangledown q_h\|_{L^{\frac{m+1}{m}}}, \quad \forall q_h \in M_h.$$

Now, taking into account the linearity of the map $q_h \mapsto -\int_\Omega b q_h d\boldsymbol{x} + \int_\Gamma g q_h d\sigma$ and Proposition 3.1, there exists a unique $\boldsymbol{u}_{h,\ell} \in X_h/V_h$ such that ,

$$\int_\Omega \boldsymbol{u}_{h,\ell} \cdot \bigtriangledown q_h d\boldsymbol{x} = -\int_\Omega b q_h d\boldsymbol{x} + \int_\Gamma g q_h d\sigma, \forall q_h \in M_h. \tag{3.13}$$

Consequently,

$$\int_\Omega \bigtriangledown q_h \cdot \boldsymbol{u}_{h,\ell} d\boldsymbol{x} \leq \|b\|_{L^{\frac{d(m+1)}{d+(m+1)}}(\Omega)} \|q_h\|_{L^{\frac{d(m+1)}{m(d-1)-1}}\Omega} + \|g\|_{L^{\frac{(d-1)(m+1)}{d}}(\Gamma)} \|q_h\|_{L^{\frac{(d-1)(m+1)}{m(d-1)-1}}(\Gamma)}$$

In view of the discrete Sobolev embeddings for regular mesh in [31, Propositions 4 and 5],

$$\sum_{T\in\mathcal{T}_h}\int_T \bigtriangledown q_h \cdot \boldsymbol{u}_{h,\ell} d\boldsymbol{x} \leq C_{b,g}\left(\sum_{T\in\mathcal{T}_h}\| \bigtriangledown q_h\|^{\frac{m+1}{m}}_{L^{\frac{m+1}{m}}(T)}\right)^{\frac{m}{m+1}},$$

$C_{b,g} = \|b\|_{L^{\frac{d(m+1)}{d+(m+1)}}(\Omega)} + \|g\|_{L^{\frac{(d-1)(m+1)}{d}}(\Gamma)}$. Hence, applying Proposition 3.1 to the above inequality, we get the following bound,

$$\|\boldsymbol{u}_{h,\ell}\|_{L^{m+1}/V_h} \leq C\, C_{b,g}. \tag{3.14}$$

Splitting $\boldsymbol{u}_h = \boldsymbol{u}_{h,0} + \boldsymbol{u}_{h,\ell}$, where $\boldsymbol{u}_{h,0} \in V_h$ is the solution of

$$\int_\Omega \mathcal{A}(\boldsymbol{u}_{h,0} + \boldsymbol{u}_{h,\ell}) \cdot \boldsymbol{\varphi}_h d\boldsymbol{x} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi} d\boldsymbol{x}, \forall \boldsymbol{\varphi}_h \in V_h, \tag{3.15}$$

that is, $\boldsymbol{u}_{h,0}$ is the discrete version of $\boldsymbol{u}_0$ defined by (2.22).

Proposition 3.2 *Formulation* (3.15) *is equivalent to* (3.9)-(3.10).

Proof. Obviously, (3.9)-(3.10) implies (3.15).

Conversely, suppose $\boldsymbol{u}_{h,0} \in V_h$ is the solution of (3.15). It follows that $\boldsymbol{f} - \mathcal{A}(\boldsymbol{u}_{h,0} + \boldsymbol{u}_{h,\ell}) \in V_h^{\perp}$. Therefore, the discrete inf-sup condition in Proposition 3.1 and the extension of Babuska - Brezzi theory to reflexive Banach spaces [5] imply the existence and uniqueness of $P_h \in M_h$, satisfying

$$\boldsymbol{f} - \mathcal{A}(\boldsymbol{u}_{h,0} + \boldsymbol{u}_{h,\ell}) = \bigtriangledown p_h. \tag{3.16}$$

Then, (3.9) follows from multiplying the last equation with $\boldsymbol{\varphi}_h \in X$ and integrating over Ω. Observe that $\int_\Omega \nabla q \cdot \boldsymbol{u}_h d\boldsymbol{x} = \int_\Omega \nabla q \cdot \boldsymbol{u}_{h,\ell} d\boldsymbol{x}$, since $\boldsymbol{u}_{h,0} \in V_h$. Then, in view of (3.13), (3.10) is satisfied. ∎

Theorem 3.1 *The problem* (3.9)-(3.10) *has a unique solution* $(\boldsymbol{u}_h, p_h) \in X_h \times M_h$. *Moreover, For any* $\boldsymbol{u}_{h,\ell} \in X_h$ *satisfying* (3.14), *the following bounds hold:*

$$\|\boldsymbol{u}_h\|_{[L^{m+1}(\Omega)]^d} \leq \left(\|\boldsymbol{u}_{h,\ell}\|^{m+1}_{[L^{m+1}(\Omega)]^d} + C\frac{\mu}{\rho} \frac{\|\boldsymbol{K}^{-1}\|_{L^\infty}}{\lambda_{min}} \|\boldsymbol{u}_{h,\ell}\|^2_{L^2} \right)^{\frac{1}{m+1}}, \tag{3.17}$$

and

$$\| \nabla p_h\|_{[L^{\frac{m+1}{m}}(\Omega)]^d} \leq \frac{\mu}{\rho}\|\boldsymbol{K}^{-1}\|_{L^\infty}\|\boldsymbol{u}_h\|_{[L^{\frac{m+1}{m}}(\Omega)]^d} + \frac{\beta}{\rho}\|\boldsymbol{u}_h\|^2_{[L^{m+1}(\Omega)]^d} + \|\boldsymbol{f}\|_{[L^{\frac{m+1}{m}}(\Omega)]^d}. \tag{3.18}$$

Proof. In view of the Brezzi Babuska theory in [5], the existence and uniqueness $\boldsymbol{u}_h \in X_h$ and $p_h \in M_h$ follows from the inf-sup condition in Proposition 3.1 and the properties of $\mathcal{A}$ itemized in Chapter 3. The proof of Estimate (3.17) follows from the discrete argument for the proof of (2.30), see [3]. Estimate (3.18) is a consequence of equation (3.16). ∎

3.2 Convergence of the Discrete solution

In this section, we give the convergence analysis of $\{\boldsymbol{u}_h\}$ and $\{p_h\}$ in X and $W^{1,\frac{m+1}{m}}(\Omega)$, respectively.

Theorem 3.2 *Let* $\boldsymbol{u}$ *be the solution of* (2.22) *and let* $\boldsymbol{u}_h$ *be the solution of* (3.15).

Then,

$$\lim_{h\to 0} \boldsymbol{u}_h = \boldsymbol{u} \text{ weakly in } X.$$

Proof. From the uniform bounds (3.17) and (3.18), we can extract from $\{\boldsymbol{u}_h\}$, $\{p_h\}$ subsequences still denoted by $\{\boldsymbol{u}_h\}$, $\{p_h\}$ that converge weakly in X and $L^{\frac{m+1}{m}}(\Omega)$, respectively. Suppose $\{\boldsymbol{u}_h\}$ converges to some function $\boldsymbol{u} \in X$. It therefore suffices to prove that the weak limit $\boldsymbol{u}$ satisfies (2.22). To see this, we split $\boldsymbol{u}_h$ as $\boldsymbol{u}_{h,0} + \boldsymbol{u}_{h,\ell}$ and take $\boldsymbol{u}_{h,\ell} = \Pi_h \boldsymbol{u}_\ell$. Then, in view of (2.4),

$$\int_\Omega (\mathcal{A}(\boldsymbol{u}_{h,0} + \boldsymbol{u}_{h,\ell}) - \mathcal{A}(\boldsymbol{v}_h + \boldsymbol{u}_{h,\ell})) \cdot (\boldsymbol{u}_{h,0} - \boldsymbol{v}_h)\, d\boldsymbol{x} \geq 0, \forall \boldsymbol{v}_h \in V_h.$$

According to (3.15),

$$\int_\Omega \mathcal{A}(\boldsymbol{u}_{h,0} + \boldsymbol{u}_{h,\ell}) \cdot (\boldsymbol{u}_{h,0} - \boldsymbol{v}_h)\, d\boldsymbol{x} = \int_\Omega \boldsymbol{f} \cdot (\boldsymbol{u}_{h,0} - \boldsymbol{v}_h)\, d\boldsymbol{x}, \forall \boldsymbol{v}_h \in V_h.$$

Therefore,

$$\int_\Omega (\mathcal{A}(\boldsymbol{v}_h + \boldsymbol{u}_{h,\ell}) - \boldsymbol{f}) \cdot (\boldsymbol{u}_{h,0} - \boldsymbol{v}_h)\, d\boldsymbol{x} \leq 0, \quad \forall \boldsymbol{v}_h \in V_h. \tag{3.19}$$

Now set $\boldsymbol{v}_h = \Pi_h \boldsymbol{v}$ for an arbitrary $\boldsymbol{v} \in V$. Then, $\Pi_h \boldsymbol{v} \in V_h$ and by (3.3), $\Pi_h \boldsymbol{v}$ converges strongly to $\boldsymbol{v}$ in V.

Claim 3.1

$$\text{i.)} \lim_{h\to 0} \mathcal{A}(\Pi_h \boldsymbol{v} + \boldsymbol{u}_{h,\ell}) = \mathcal{A}(\boldsymbol{v} + \boldsymbol{u}_\ell) \quad \text{strongly in} \quad [L^{\frac{m+1}{m}}(\Omega)]^d.$$

$$\text{ii.)} \lim_{h\to 0} (\boldsymbol{u}_{h,0} - \Pi_h \boldsymbol{v}) = \boldsymbol{u}_0 - \boldsymbol{v} \quad \text{weakly in} \quad X.$$

Proof. By (2.31),

$$\begin{aligned} &\|\mathcal{A}(\Pi_h \boldsymbol{v} + \boldsymbol{u}_{h,\ell}) - \mathcal{A}(\boldsymbol{v} + \boldsymbol{u}_\ell)\|_{L^{\frac{m+1}{m}}} \\ &\le \left(\int_\Omega \left[|\Pi_h \boldsymbol{v} + \boldsymbol{u}_{h,\ell} - (\boldsymbol{v} + \boldsymbol{u}_\ell)| \left(\frac{\mu}{\rho} \|\boldsymbol{K}^{-1}\|_{L^\infty} + \frac{\beta}{\rho} m 2^{m-2} \chi \right) \right]^{\frac{m+1}{m}} \right)^{\frac{m}{m+1}}, \end{aligned}$$

where $\chi = |\Pi_h \boldsymbol{v} + \boldsymbol{u}_{h,\ell}|^{m-1} + |\boldsymbol{v} + \boldsymbol{u}_\ell|^{m-1}$. In view of (3.3), we deduce from the last expression that

$$\|\mathcal{A}(\Pi_h \boldsymbol{v} + \boldsymbol{u}_{h,\ell}) - \mathcal{A}(\boldsymbol{v} + \boldsymbol{u}_\ell)\|_{L^{\frac{m+1}{m}}} \longrightarrow 0 \quad \text{as} \quad h \longrightarrow 0,$$

and hence i.) is established.

For the proof of ii.), observe that for any $\boldsymbol{v} \in X$, $\mathcal{A}(\boldsymbol{v}) \in [L^{\frac{m+1}{m}}(\Omega)]^d$. Therefore,

$$\begin{aligned} &\left| \langle \mathcal{A}(v), \boldsymbol{u}_{h,0} - \Pi_h \boldsymbol{v} \rangle_{L^{\frac{m+1}{m}} \times L^{m+1}} - \langle \mathcal{A}(v), \boldsymbol{u}_0 - \boldsymbol{v} \rangle_{L^{\frac{m+1}{m}} \times L^{m+1}} \right| \\ &= \left| \langle \mathcal{A}(v),\ \boldsymbol{u}_{h,0} - \Pi_h \boldsymbol{v} - (\boldsymbol{u}_0 - \boldsymbol{v}) \rangle_{L^{\frac{m+1}{m}} \times L^{m+1}} \right| \\ &\le \|\mathcal{A}(v)\|_{[L^{\frac{m+1}{m}}(\Omega)]^d} \|\boldsymbol{u}_{h,0} - \Pi_h \boldsymbol{v} - (\boldsymbol{u}_0 - \boldsymbol{v})\|_{[L^{m+1}(\Omega)]^d} \longrightarrow 0, \quad \text{as} \quad h \longrightarrow 0. \end{aligned}$$

Hence, the claim is proved. ∎

Now, passing to the limit in (3.19), we get,

$$\int_\Omega (\mathcal{A}(\boldsymbol{v}+\boldsymbol{u}_\ell)-\boldsymbol{f})\cdot(\boldsymbol{u}_0-\boldsymbol{v})\,d\boldsymbol{x}\leq 0,\quad \forall \boldsymbol{v}\in V.$$

Let $\boldsymbol{w}\in V$ and set $\boldsymbol{v}=\boldsymbol{u}_0+\lambda\boldsymbol{w},\lambda>0$. Then,

$$-\lambda\int_\Omega (\mathcal{A}(\boldsymbol{u}_0+\lambda\boldsymbol{w}+\boldsymbol{u}_\ell)-\boldsymbol{f})\cdot\boldsymbol{w}d\boldsymbol{x}\leq 0\ \ \forall \boldsymbol{w}\in V,$$

which implies that

$$\int_\Omega (\mathcal{A}(\boldsymbol{u}_0+\lambda\boldsymbol{w}+\boldsymbol{u}_\ell)-\boldsymbol{f})\cdot\boldsymbol{w}d\boldsymbol{x}\geq 0\ \ \forall \boldsymbol{w}\in V.$$

Thanks to the hemicontinuity property of $\mathcal{A}$ in (2.3), we have

$$\int_\Omega (\mathcal{A}(\boldsymbol{u}_0+\boldsymbol{u}_\ell)-\boldsymbol{f})\cdot\boldsymbol{w}d\boldsymbol{x}\geq 0,\ \forall \boldsymbol{w}\in V.$$

Taking $\boldsymbol{v}=\boldsymbol{u}_0-\lambda\boldsymbol{w}$ and repeating thesame calculation, we obtain

$$\int_\Omega (\mathcal{A}(\boldsymbol{u}_0+\boldsymbol{u}_\ell)-\boldsymbol{f})\cdot\boldsymbol{w}d\boldsymbol{x}\leq 0,\ \forall \boldsymbol{w}\in V.$$

Thus, we get

$$\int_\Omega (\mathcal{A}(\boldsymbol{u}_0+\boldsymbol{u}_\ell)-\boldsymbol{f})\cdot\boldsymbol{w}d\boldsymbol{x}=0,\ \forall \boldsymbol{w}\in V,$$

which implies that

$$\int_\Omega \mathcal{A}(\boldsymbol{u}_0 + \boldsymbol{u}_\ell) \cdot \boldsymbol{w} d\boldsymbol{x} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{w} d\boldsymbol{x}, \ \forall \boldsymbol{w} \in V. \tag{3.20}$$

∎

Theorem 3.3 *The discrete solution $\boldsymbol{u}_h$ converges strongly to $\boldsymbol{u}$ in $[L^m(\Omega)]^d$.*

Proof. Set $\boldsymbol{\varphi}_h = \boldsymbol{v}_h$ in (3.9) and subtract from (2.1);

$$\int_\Omega (\mathcal{A}(\boldsymbol{u}_{h,0} + \boldsymbol{u}_{h,\ell}) - \mathcal{A}(\boldsymbol{u}_0 + \boldsymbol{u}_\ell)) \cdot \boldsymbol{v}_h d\boldsymbol{x} + \int_\Omega \bigtriangledown(p_h - p) \cdot \boldsymbol{v}_h d\boldsymbol{x} = 0, \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h.$$

Since p_h, $I_h p \in M_h$ and since $\boldsymbol{v}_h \in V_h$, last expression is equivalent to

$$\int_\Omega (\mathcal{A}(\boldsymbol{u}_{h,0} + \boldsymbol{u}_{h,\ell}) - \mathcal{A}(\boldsymbol{u}_0 + \boldsymbol{u}_\ell)) \cdot \boldsymbol{v}_h d\boldsymbol{x} = \int_\Omega \bigtriangledown(p - I_h p) \cdot \boldsymbol{v}_h d\boldsymbol{x}, \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h.$$

Inserting $\mathcal{A}(\Pi_h(\boldsymbol{u}_0 + \boldsymbol{u}_\ell))$ and taking $\boldsymbol{v}_h = \boldsymbol{u}_h - \Pi_h \boldsymbol{u} = \boldsymbol{u}_{h,0} - \Pi_h \boldsymbol{u}_0$, give

$$\begin{aligned}
&\int_\Omega (\mathcal{A}(\boldsymbol{u}_{h,0} + \boldsymbol{u}_{h,\ell}) - \mathcal{A}(\Pi_h(\boldsymbol{u}_0 + \boldsymbol{u}_\ell))) \cdot (\boldsymbol{u}_{h,0} - \Pi_h \boldsymbol{u}_0) d\boldsymbol{x} \\
&= - \int_\Omega (\mathcal{A}(\Pi_h(\boldsymbol{u}_0 + \boldsymbol{u}_\ell)) - \mathcal{A}(\boldsymbol{u}_0 + \boldsymbol{u}_\ell)) \cdot (\boldsymbol{u}_{h,0} - \Pi_h \boldsymbol{u}_0) d\boldsymbol{x} \\
&\quad + \int_\Omega \bigtriangledown(p - I_h p) \cdot (\boldsymbol{u}_{h,0} - \Pi_h \boldsymbol{u}_0) d\boldsymbol{x}.
\end{aligned} \tag{3.21}$$

In view of the monotonicity property (2.4) of $\mathcal{A}$, we get,

$$\begin{aligned}
&\frac{\mu}{\rho}\lambda_{min}\|\boldsymbol{u}_h - \Pi_h u\|^2_{[L^m(\Omega)]^d} \\
&\leq \int_\Omega \bigtriangledown(p - I_h p)\cdot(\boldsymbol{u}_{h,0} - \Pi_h \boldsymbol{u}_0)d\boldsymbol{x} \\
&\qquad - \int_\Omega (\mathcal{A}(\Pi_h(\boldsymbol{u}_0 + \boldsymbol{u}_\ell)) - \mathcal{A}(\boldsymbol{u}_0 + \boldsymbol{u}_\ell))\cdot(\boldsymbol{u}_{h,0} - \Pi_h \boldsymbol{u}_0)d\boldsymbol{x}. \qquad (3.22)
\end{aligned}$$

Using (3.5), (3.3) and (2.31),

$$\lim_{h\to 0} \mathcal{A}(\Pi_h(\boldsymbol{u}_0 + \boldsymbol{u}_\ell) - \mathcal{A}(\boldsymbol{u}_0 + \boldsymbol{u}_\ell)) = 0 \;\text{ strongly in }\; [L^{\frac{m+1}{m}}(\Omega)]^d. \qquad (3.23)$$

Passing to the limit in (3.22) and using (3.5), (3.23) and Theorem 3.2 we establish the strong convergence of $\boldsymbol{u}_h$ in $[L^m(\Omega)]^d$. ▌

The bound in (3.18) shows that the sequence $\{\bigtriangledown p_h\}$ is uniformly bounded in the reflexive space $[L^{\frac{m+1}{m}}(\Omega)]^d$. Consequently, $\{p_h\}$ is uniformly bounded in $W^{1,\frac{m+1}{m}}(\Omega)$ because $\|p_h\|_{W^{1,\frac{m+1}{m}}(\Omega)} \leq C\| \bigtriangledown p_h\|_{[L^{\frac{m+1}{m}}(\Omega)]^d}$. Therefore, there exists $q \in W^{1,\frac{m+1}{m}}(\Omega)$ such that $\lim_{h\to 0} p_h = q$ weakly in $W^{1,\frac{m+1}{m}}(\Omega)$.

Next, we show that q is the second component of the solution of (2.1)-(2.2).

Proposition 3.3 *If* $(\boldsymbol{u}, p)$ *is the solution of* (2.1)-(2.2), *then,* $\lim_{h\to 0} p_h = p$ weakly in $W^{1,\frac{m+1}{m}}(\Omega)$.

Proof. Consider (3.9), with $\boldsymbol{\varphi}_h = \Pi_h \boldsymbol{v}$, to get

$$\frac{\mu}{\rho}\int_\Omega (\boldsymbol{K}^{-1}\boldsymbol{u}_h)\cdot \Pi_h \boldsymbol{v} d\boldsymbol{x} + \frac{\beta}{\rho}\int_\Omega |\boldsymbol{u}_h|^{m-1}\boldsymbol{u}_h \cdot \Pi_h \boldsymbol{v} d\boldsymbol{x} + \int_\Omega \bigtriangledown p_h \cdot \Pi_h \boldsymbol{v} d\boldsymbol{x} = 0.$$

Equivalently,

$$\int_\Omega \mathcal{A}(\boldsymbol{u}_h) \cdot \Pi_h \boldsymbol{v} d\boldsymbol{x} + \int_\Omega \bigtriangledown p_h \cdot \Pi_h \boldsymbol{v} d\boldsymbol{x} = 0.$$

Inserting $\mathcal{A}(\boldsymbol{u})$ and re-arranging, we reach

$$\int_\Omega \bigtriangledown p_h \cdot \Pi_h \boldsymbol{v} d\boldsymbol{x} = -\int_\Omega (\mathcal{A}(\boldsymbol{u}_h) - \mathcal{A}(\boldsymbol{u})) \cdot \Pi_h \boldsymbol{v} d\boldsymbol{x} - \int_\Omega \mathcal{A}(\boldsymbol{u}) \cdot \Pi_h \boldsymbol{v} d\boldsymbol{x}. \quad (3.24)$$

In view of (2.31), we get

$$\begin{aligned}
&\left| \int_\Omega (\mathcal{A}(\boldsymbol{u}_h) - \mathcal{A}(\boldsymbol{u})) \cdot \Pi_h \boldsymbol{v} d\boldsymbol{x} \right| \\
&\leq \|\boldsymbol{u}_h - \boldsymbol{u}\|_{[L^m(\Omega)]^d} \frac{\mu}{\rho} \|\boldsymbol{K}^{-1}\|_{L^\infty} \|\Pi_h \boldsymbol{v}\|_{[L^{m_1}(\Omega)]^d} \\
&\quad + \frac{\beta}{\rho} m 2^{m-2} \|\boldsymbol{u}_h - \boldsymbol{u}\|_{[L^m(\Omega)]^d} (\||\boldsymbol{u}_h|^{m-1}\|_{[L^{2m_1}(\Omega)]^d} + \||\boldsymbol{u}|^{m-1}\|_{[L^{2m_1}(\Omega)]^d}) \|\Pi_h \boldsymbol{v}\|_{[L^{2m_1}(\Omega)]^d},
\end{aligned}$$

$m_1 = \frac{m}{m-1}$. The strong convergence of $\boldsymbol{u}_h \in [L^m(\Omega)]^d$ and $\Pi_h \boldsymbol{v} \in L^r(\Omega)$ for $1 \leq r \leq \infty$ imply that

$$\lim_{h \to 0} \int_\Omega (\mathcal{A}(\boldsymbol{u}_h) - \mathcal{A}(\boldsymbol{u})) \cdot \Pi_h \boldsymbol{v} \, d\boldsymbol{x} = 0.$$

Finally, passing to the limit to both sides of (3.24) and then comparing with (2.1), we deduce that $q = p$. ▮

Proposition 3.4 $\boldsymbol{u}_h$ *converges strongly to* $\boldsymbol{u}$ *in* X

Proof. Choose $\boldsymbol{\varphi}_h = \boldsymbol{u}_h = \boldsymbol{u}_{h,0} + \boldsymbol{u}_{h,\ell}$ in (3.9)

$$\frac{\mu}{\rho}\int_\Omega \boldsymbol{K}^{-1}\boldsymbol{u}_h \cdot \boldsymbol{u}_h d\boldsymbol{x} + \frac{\beta}{\rho}\int_\Omega |\boldsymbol{u}_h|^{m+1} d\boldsymbol{x} + \int_\Omega \triangledown p_h \cdot \boldsymbol{u}_{h,0} d\boldsymbol{x} + \int_\Omega \triangledown p_h \cdot \boldsymbol{u}_{h,\ell} d\boldsymbol{x} = 0.$$

Since $\boldsymbol{u}_{h,0} \in V_h$, $\displaystyle\int_\Omega \triangledown p_h \cdot \boldsymbol{u}_{h,0} d\boldsymbol{x} = 0$, therefore,

$$\frac{\mu}{\rho}\int_\Omega \boldsymbol{K}^{-1}\boldsymbol{u}_h \cdot \boldsymbol{u}_h d\boldsymbol{x} + \frac{\beta}{\rho}\int_\Omega |\boldsymbol{u}_h|^{m+1} d\boldsymbol{x} + \int_\Omega \triangledown p_h \cdot \boldsymbol{u}_{h,\ell} d\boldsymbol{x} = 0. \tag{3.25}$$

In view of the strong convergence of $\boldsymbol{u}_h$ to $\boldsymbol{u}$ in $[L^m(\Omega)]^d$ and $\boldsymbol{u}_{h,\ell}$ to $\boldsymbol{u}_\ell$ in $[L^m(\Omega)]^d$, and the weak convergence of $\triangledown p_h$ to $\triangledown p$ in $[L^{\frac{m+1}{m}}(\Omega)]^d$. Passing to the limit in (3.25) and using $\boldsymbol{u} = \boldsymbol{u}_0 + \boldsymbol{u}_\ell$ results to

$$\frac{\mu}{\rho}\int_\Omega \boldsymbol{K}^{-1}\boldsymbol{u} \cdot \boldsymbol{u} d\boldsymbol{x} + \frac{\beta}{\rho}\lim_{h\to 0}\int_\Omega |\boldsymbol{u}_h|^{m+1} d\boldsymbol{x} + \int_\Omega \triangledown p \cdot \boldsymbol{u} d\boldsymbol{x} = 0. \tag{3.26}$$

Comparing with (2.1) and taking $\boldsymbol{u}$ as a test function will complete the proof. ▌

Next, we prove the strong convergence of $\triangledown p_h$ to $\triangledown p$ in $[L^{\frac{m+1}{m}}(\Omega)]^d$.

Theorem 3.4 $\lim_{h\to 0} p_h = p$ *strongly in* $W^{1,\frac{m+1}{m}}(\Omega)$.

Proof. Subtracting (2.1) from (3.9) and inserting $I_h p$ gives

$$\sum_{T\in\mathbb{T}_h}\int_T \triangledown(p - p_h)\cdot\boldsymbol{\varphi}_h d\boldsymbol{x} = \int_\Omega (\mathcal{A}(\boldsymbol{u}_h) - \mathcal{A}(\boldsymbol{u}))\cdot\boldsymbol{\varphi}_h d\boldsymbol{x},\ \forall\boldsymbol{\varphi}_h \in X_h.$$

Hence

$$\left|\sum_{T\in\mathbb{T}_h}\int_T \triangledown(I_hp-p_h)\cdot\boldsymbol{\varphi}_h d\boldsymbol{x}\right| \le \int_\Omega |(\mathcal{A}(\boldsymbol{u}_h)-\mathcal{A}(\boldsymbol{u}))\cdot\boldsymbol{\varphi}_h|\, d\boldsymbol{x} + \sum_{T\in\mathbb{T}_h}\int_T |\triangledown(p-I_hp)\cdot\boldsymbol{\varphi}_h| d\boldsymbol{x}$$

Applying (2.31) and using $\|\boldsymbol{\varphi}_h\|_{[L^{m_1}(\Omega)]^d} \le |\Omega|^{1-m_2}\|\boldsymbol{\varphi}_h\|_{[L^{m+1}(\Omega)]^d}$, we get

$$\begin{aligned}
&\int_\Omega \Big|(\mathcal{A}(\boldsymbol{u}_h)-\mathcal{A}(\boldsymbol{u}))\cdot\boldsymbol{\varphi}_h\Big| d\boldsymbol{x}\\
&\le \|\boldsymbol{u}_h-\boldsymbol{u}\|_{[L^m(\Omega)]^d}\|\boldsymbol{\varphi}_h\|_{[L^{m+1}(\Omega)]^d}\\
&\times\Big(\frac{\mu}{\rho}\|\boldsymbol{K}^{-1}\|_{L^\infty}|\Omega|^{1-m_2}+\frac{\beta}{\rho}m2^{m-2}(\||\boldsymbol{u}_h|^{m-1}\|_{[L^{\bar m}(\Omega)]^d}+\||\boldsymbol{u}|^{m-1}\|_{[L^{\bar m}(\Omega)]^d})\Big),
\end{aligned}$$

where $m_2 = \frac{m}{(m-1)(m+1)}$ and $\bar m = \frac{m(m+1)}{m^2-m-1}$.

Dividing through by $\|\boldsymbol{\varphi}_h\|_{[L^{m+1}(\Omega)]^d}$ for nonzero $\boldsymbol{\varphi}_h\in X_h$ and using

$$\sum_{T\in\mathbb{T}_h}\int_T |\triangledown(p-I_hp)\cdot\boldsymbol{\varphi}_h| d\boldsymbol{x} \le \|\triangledown(p-I_hp)\|_{[L^{\frac{m+1}{m}}(\Omega)]^d}\|\boldsymbol{\varphi}_h\|_{[L^{m+1}(\Omega)]^d},$$

leads to

$$\begin{aligned}
&\frac{1}{\|\boldsymbol{\varphi}_h\|_{[L^{m+1}(\Omega)]^d}}\left|\sum_{T\in\mathbb{T}_h}\int_T \triangledown(I_hp-p_h)\cdot\boldsymbol{\varphi}_h d\boldsymbol{x}\right|\\
&\le \|\boldsymbol{u}_h-\boldsymbol{u}\|_{[L^m(\Omega)]^d}\Big(\frac{\mu}{\rho}\|\boldsymbol{K}^{-1}\|_{L^\infty}|\Omega|^{1-m_2}+\frac{\beta}{\rho}m2^{m-2}(\||\boldsymbol{u}_h|^{m-1}\|_{[L^{\bar m}(\Omega)]^d}+\||\boldsymbol{u}|^{m-1}\|_{[L^{\bar m}(\Omega)]^d})\Big)\\
&\quad+\|\triangledown(p-I_hp)\|_{[L^{\frac{m+1}{m}}(\Omega)]^d}.
\end{aligned}$$

Applying the discrete inf-sup condition in Proposition 3.1, we arrive at

$$
\begin{aligned}
\| \bigtriangledown (I_h p - p_h)\|_{[L^{\frac{m+1}{m}}(\Omega)]^d} \leq\ & \frac{\mu}{\rho}|\Omega|^{1-m_2}\|\boldsymbol{K}^{-1}\|_{L^\infty}\|\boldsymbol{u}_h - \boldsymbol{u}\|_{[L^m(\Omega)]^d} \\
& + \frac{\beta}{\rho} m 2^{m-2}\|\boldsymbol{u}_h - \boldsymbol{u}\|_{[L^m(\Omega)]^d} \\
& \times \left(\||\boldsymbol{u}_h|^{m-1}\|_{[L^{\bar{m}}(\Omega)]^d} + \||\boldsymbol{u}|^{m-1}\|_{[L^{\bar{m}}(\Omega)]^d}\right) \\
& + \| \bigtriangledown (p - I_h p)\|_{[L^{\frac{m+1}{m}}(\Omega)]^d} \qquad (3.27)
\end{aligned}
$$

Since $\||\psi|^{m-1}\|_{[L^{\bar{m}}(\Omega)]^d} = \|\psi\|^{m-1}_{[L^{\bar{m}(m-1)}(\Omega)]^d}$, from the decomposition $p - p_h = (p_h - I_h p) + (I_h p - p)$ and the above inequality, we obtain

$$
\begin{aligned}
\| \bigtriangledown (p - p_h)\|_{[L^{\frac{m+1}{m}}(\Omega)]^d} \leq\ & \frac{\mu}{\rho}|\Omega|^{1-m_2}\|\boldsymbol{K}^{-1}\|_{L^\infty}\|\boldsymbol{u}_h - \boldsymbol{u}\|_{[L^m(\Omega)]^d} \\
& + \frac{\beta}{\rho} m 2^{m-2}\|\boldsymbol{u} - \boldsymbol{u}_h\|_{[L^m(\Omega)]^d} \\
& \times \left(\|\boldsymbol{u}_h\|^{m-1}_{[L^{\bar{m}(m-1)}(\Omega)]^d} + \|\boldsymbol{u}\|^{m-1}_{[L^{\bar{m}(m-1)}(\Omega)]^d}\right) \\
& + 2\| \bigtriangledown (p - I_h p)\|_{[L^{\frac{m+1}{m}}(\Omega)]^d} .
\end{aligned}
$$

Thanks to the convergence of $\boldsymbol{u}_h$ and $I_h p$, the theorem is thus established. ▮

3.3 Error estimates

In this section, we present the priori error estimates for both the velocity and pressure.

3.3.1 Error estimates for velocity

Theorem 3.5 *Let* $(\boldsymbol{u}, p)$ *solve* (2.1)-(2.2), $(\boldsymbol{u}_h, p_h)$ *solve* (3.9)–(3.10), *and assume that* $\boldsymbol{u} \in [L^{2m}(\Omega)]^d$. *Then for any arbitrary element* $\boldsymbol{w}_h \in X_h$, *the following estimate holds,*

$$\begin{aligned}\|\boldsymbol{u}-\boldsymbol{u}_h\|_{[L^m(\Omega)]^d} \leq & \left(C + \frac{\|\boldsymbol{K}^{-1}\|_{L^\infty}}{\lambda_{min}}\right)\|\boldsymbol{u}-\boldsymbol{w}_h\|_{[L^{m_1}(\Omega)]^d} \\ & + \frac{\beta}{\mu\lambda_{min}} m 2^{m-2}\|\boldsymbol{u}-\boldsymbol{w}_h\|_{[L^{2m_1}(\Omega)]^d}\left(\|\boldsymbol{w}_h\|^{m-1}_{[L^{2m}(\Omega)]^d} + \|\boldsymbol{u}\|^{m-1}_{[L^{2m}(\Omega)]^d}\right) \\ & + \frac{\rho}{\mu\lambda_{min}}\| \bigtriangledown (p - I_h p)\|_{[L^{m_1}(\Omega)]^d} \,. \end{aligned} \tag{3.28}$$

Proof. Taking the difference between (3.9) and (2.1),

$$\int_\Omega (\mathcal{A}(\boldsymbol{u}_h) - \mathcal{A}(\boldsymbol{u}))\cdot \boldsymbol{\varphi}_h d\boldsymbol{x} + \int_\Omega \bigtriangledown(p_h - p)\cdot\boldsymbol{\varphi}_h d\boldsymbol{x} = 0, \quad \forall \boldsymbol{\varphi}_h \in V_h.$$

This expression can also be written as, $\forall \boldsymbol{\varphi}_h \in V_h$, $\forall \mu \in M_h$,

$$\int_\Omega (\mathcal{A}(\boldsymbol{u}_h) - \mathcal{A}(\boldsymbol{u}))\cdot \boldsymbol{\varphi}_h d\boldsymbol{x} = \int_\Omega \bigtriangledown(p - \mu_h)\cdot\boldsymbol{\varphi}_h d\boldsymbol{x}.$$

Inserting $\mathcal{A}(\boldsymbol{w}_h)$, taking $\mu_h = I_h p$ and choosing $\boldsymbol{\varphi}_h = \boldsymbol{u}_h - \boldsymbol{w}_h$, we get

$$\begin{aligned}\int_\Omega (\mathcal{A}(\boldsymbol{u}_h) - \mathcal{A}(\boldsymbol{w}_h))\cdot(\boldsymbol{u}_h - \boldsymbol{w}_h) d\boldsymbol{x} = & -\int_\Omega (\mathcal{A}(\boldsymbol{w}_h) - \mathcal{A}(\boldsymbol{u}))\cdot(\boldsymbol{u}_h - \boldsymbol{w}_h) d\boldsymbol{x} \\ & + \int_\Omega \bigtriangledown(p - I_h p)\cdot(\boldsymbol{u}_h - \boldsymbol{w}_h)\cdot d\boldsymbol{x}\,. \end{aligned}$$

Applying (2.4) and (2.31) to the LHS and RHS, respectively,

$$
\begin{aligned}
\frac{\mu}{\rho}\lambda_{min}\|\boldsymbol{u}_h-\boldsymbol{w}_h\|^2_{[L^m(\Omega)]^d} \le & \int_\Omega \frac{\mu}{\rho}\|\boldsymbol{K}^{-1}\|_{L^\infty}|\boldsymbol{w}_h-\boldsymbol{u}||\boldsymbol{u}_h-\boldsymbol{w}_h|d\boldsymbol{x} \\
& + \int_\Omega \frac{\beta}{\rho} m2^{m-2}|\boldsymbol{w}_h-\boldsymbol{u}|\left(|\boldsymbol{w}_h|^{m-1}+|\boldsymbol{u}|^{m-1}\right)|\boldsymbol{u}_h-\boldsymbol{w}_h|d\boldsymbol{x} \\
& + \int_\Omega |\bigtriangledown(p-I_hp)||\boldsymbol{u}_h-\boldsymbol{w}_h|d\boldsymbol{x},
\end{aligned}
$$

and so,

$$
\begin{aligned}
\frac{\mu}{\rho}\lambda_{min}\|\boldsymbol{u}_h-\boldsymbol{w}_h\|^2_{[L^m(\Omega)]^d} \le \|\boldsymbol{u}_h-\boldsymbol{w}_h\|_{[L^m(\Omega)]^d}\Bigg(& \frac{\mu}{\rho}\|\boldsymbol{K}^{-1}\|_{L^\infty}\|\boldsymbol{w}_h-\boldsymbol{u}\|_{[L^{m_1}(\Omega)]^d} \\
& + \frac{\beta}{\rho} m2^{m-2}\|\boldsymbol{w}_h-\boldsymbol{u}\|_{[L^{2m_1}(\Omega)]^d}\Big(\||\boldsymbol{w}_h|^{m-1}\|_{[L^{2m_1}(\Omega)]^d} \\
& + \||\boldsymbol{u}|^{m-1}\|_{[L^{2m_1}(\Omega)]^d}\Big) + \|\bigtriangledown(p-I_hp)\|_{[L^{m_1}(\Omega)]^d}\Bigg).
\end{aligned}
$$

Using the identity $\||\psi|^{m-1}\|_{[L^{2m_1}(\Omega)]^d} = \|\psi\|^{m-1}_{[L^{2m}(\Omega)]^d}$, we deduce

$$
\begin{aligned}
\|\boldsymbol{u}_h-\boldsymbol{w}_h\|_{[L^m(\Omega)]^d} \le & \frac{\|\boldsymbol{K}^{-1}\|_{L^\infty}}{\lambda_{min}}\|\boldsymbol{w}_h-\boldsymbol{u}\|_{[L^{m_1}(\Omega)]^d} \\
& + \frac{\beta}{\mu\lambda_{min}} m2^{m-2}\|\boldsymbol{w}_h-\boldsymbol{u}\|_{[L^{2m_1}(\Omega)]^d}\left(\|\boldsymbol{w}_h\|^{m-1}_{[L^{2m}(\Omega)]^d}+\|\boldsymbol{u}\|^{m-1}_{[L^{2m}(\Omega)]^d}\right) \\
& + \frac{\rho}{\mu\lambda_{min}}\|\bigtriangledown(p-I_hp)\|_{[L^{m_1}(\Omega)]^d}.
\end{aligned}
$$

Adding $\|\boldsymbol{u}-\boldsymbol{w}_h\|_{[L^m(\Omega)]^d}$ to both sides with sobolev embedding established (3.28)

∎

Remark 1 *Since* $2m_1(m-1)=2m$, *one can easily check that* $\||\boldsymbol{u}|^{m-1}\|_{[L^{2m_1}(\Omega)]^d} = \|\boldsymbol{u}\|^{m-1}_{[L^{2m}(\Omega)]^d}$ *for* $m\in(1,2]$.

Theorem 3.6 *Assume that the solution $\boldsymbol{u}$ of* (2.22) *belongs to* $[W^{1,2m_1}(\Omega)]^d$ *and let $\boldsymbol{u}_h$ be solution of* (3.15)*. Then*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{[L^m(\Omega)]^d} \leq Ch|\boldsymbol{u}|_{W^{1,2m_1}} . \tag{3.29}$$

Proof. Putting $\boldsymbol{w}_h = \Pi_h \boldsymbol{u}$ in (3.28), then applying (3.3), (3.4), and (3.5), yield the desired error estimate. ▮

3.3.2 Error estimates for pressure

From the achieved error bound in (3.28), it requires that $\boldsymbol{u}_h$ be bounded in $L^{\bar{m}(m-1)}(\Omega)$. We further assume that, there exists another constants $\tau > 0$ independent of h and T such that

$$\forall T \in \mathcal{T}_h,\ \tau h \leq\ h_T \leq \sigma \rho_T \tag{3.30}$$

With these assumptions, we can use the inverse inequality [9] on the term $\|\boldsymbol{u}_h - \Pi_h \boldsymbol{u}\|_{[L^{\bar{m}(m-1)}(\Omega)]^d}$ and the stability properties of the projection Π_h.

$$\begin{aligned}
\|\boldsymbol{u}_h\|_{[L^{\bar{m}(m-1)}(\Omega)]^d} &\leq \|\boldsymbol{u}_h - \Pi_h \boldsymbol{u}\|_{[L^{\bar{m}(m-1)}(\Omega)]^d} + \|\Pi_h \boldsymbol{u}\|_{[L^{\bar{m}(m-1)}(\Omega)]^d}, \\
&\leq Ch^{-\frac{d}{m^2-1}} \|\boldsymbol{u}_h - \Pi_h \boldsymbol{u}\|_{[L^m(\Omega)]^d} + \|\boldsymbol{u}\|_{[L^{\bar{m}(m-1)}(\Omega)]^d}, \\
&\leq C\left(h^{-\frac{d}{m^2-1}} \|\boldsymbol{u}_h - \Pi_h \boldsymbol{u}\|_{[L^m(\Omega)]^d} + \|\boldsymbol{u}\|_{[L^{\bar{m}(m-1)}(\Omega)]^d}\right).
\end{aligned} \tag{3.31}$$

Using (3.29) and Sobolev embedding, we established the Proposition below.

Proposition 3.5 *Let $\mathcal{T}_h$ be the triangulation, satisfying* (3.30)*. Let us suppose* *that the solution* $\boldsymbol{u}$ *of* (2.22) *belongs to* $[W^{1,2m_1}(\Omega)]^d$. *Then,*

$$\|\boldsymbol{u}_h\|_{[L^{\bar{m}(m-1)}(\Omega)]^d} \leq C\|\boldsymbol{u}\|_{W^{1,2m_1}}. \tag{3.32}$$

Proposition 3.6 *Let us suppose that the solution* $\boldsymbol{u}$ *of* (2.22) *belongs to* $[W^{1,2m_1}(\Omega)]^d$ *and that the solution* p *of* (2.1)–(2.2) *belongs to* $W^{2,\frac{m+1}{m}}(\Omega)$. *Then*

$$\| \nabla (p - p_h)\|_{[L^{\frac{m+1}{m}}(\Omega)]^d} \leq Ch\left(\|\boldsymbol{u}\|_{W^{1,2m_1}} + |p|_{W^{2,\frac{m+1}{m}}}\right).$$

Proof. It follows from (3.28), (3.32) and the interpolation error estimates in (3.5). ∎

3.4 Numerical Implementation

In this section, we present the solutions of the nonlinear system associated with discretization of the generalized Darcy-Forchheimer model. Define

$$X_h = span\,\{\boldsymbol{\varphi}_i\}_{i=1}^n\,, \qquad M_h = span\,\{\psi_j\}_{j=1}^r\,,$$

where $\boldsymbol{\varphi}_i$ and ψ_j are bases associated with the velocity and pressure respectively. Therefore, any $\boldsymbol{u}_h \in X_h$ and $p_h \in M_h$ can be expressed as,

$$\boldsymbol{u}_h = \sum_{i=1}^n \tilde{u}_i \boldsymbol{\varphi}_i \qquad p_h = \sum_{j=1}^r \tilde{p}_j \psi_j\ , \tag{3.33}$$

Now substituting (5.1) into equation (3.9)–(3.10), give rise to the following nonlinear system: Find $\tilde{\boldsymbol{u}} \in \mathbb{R}^n$, $\tilde{p} \in \mathbb{R}^r$ such that,

$$\begin{bmatrix} D(\tilde{\boldsymbol{u}}) & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{u}} \\ \tilde{p} \end{bmatrix} = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}, \tag{3.34}$$

where $\tilde{\boldsymbol{u}} = [\tilde{u}_1, \tilde{u}_2, \tilde{u}_3, \cdots \tilde{u}_n]^T$, $\tilde{p} = [\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \cdots \tilde{p}_m]^T$, and $D(\tilde{\boldsymbol{u}}) \in \mathbb{R}^{n\times n}$ is a diagonal matrix. Noting that the matrices $D(\tilde{\boldsymbol{u}})\tilde{\boldsymbol{u}}$ and $B \in \mathbb{R}^{r\times n}$ are corresponding to

$$\frac{\mu}{\rho}\int_\Omega \boldsymbol{K}^{-1}\boldsymbol{u}_h \cdot \boldsymbol{\varphi}_h d\boldsymbol{x} + \frac{\beta}{\rho}\int_\Omega |\boldsymbol{u}_h|^{m-1}\boldsymbol{u}_h \cdot \boldsymbol{\varphi}_h d\boldsymbol{x} \quad \text{and} \quad \int_\Omega \bigtriangledown q_h \cdot \boldsymbol{u}_h d\boldsymbol{x},$$

respectively. The matrix B^T is indeed the transpose of B. In the above system, $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$ are column vectors corresponding to,

$$\int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi}_h dx, \quad \text{and} \quad \int_\Gamma g q_h d\boldsymbol{\sigma} - \int_\Omega b q_h d\boldsymbol{x}\,,$$

respectively. The nonlinear system (5.3) is solved by Newton's method [37, 49] until the norm of the difference in successive iterates and the norm of residual were within a fixed tolerance of 10^{-6}.

CHAPTER 4

TWO-LEVEL MIXED FINITE ELEMENT METHOD FOR DARCY-FORCHEMMER EQUATION

4.1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be an open bounded subset with Lipschitz boundary Γ. Consider a steady Darcy-Forchheimer equation, describing a single phase flow of fluid through a porous medium Ω.

$$\frac{\mu}{\rho}\boldsymbol{K}^{-1}\boldsymbol{u} + \frac{\beta}{\rho}|\boldsymbol{u}|\boldsymbol{u} + \bigtriangledown p = \boldsymbol{f} \qquad \text{in} \qquad \Omega, \tag{4.1}$$

$$\mathbf{div}\boldsymbol{u} = b \qquad \text{in} \qquad \Omega, \tag{4.2}$$

$$\boldsymbol{u}\cdot\boldsymbol{n} = g \qquad \text{on} \qquad \Gamma, \tag{4.3}$$

b and g are given functions satisfying the compatibility condition:

$$\int_\Omega b(x)dx = \int_\Gamma g(\sigma)d\sigma. \tag{4.4}$$

4.2 Two-level Galekin mixed finite element algorithm

In this Chapter, we present the two-level mixed finite element method for obtaining an approximate solution to (4.1)-(4.3). It is worthy of note that the Euclidean norm $|\boldsymbol{u}|$ is not differentiable at zero. Therefore, let

$$|\boldsymbol{u}|_\epsilon = \sqrt{|\boldsymbol{u}|^2 + \epsilon^2},$$

where ϵ is a small positive constant. For ϵ very small, it is shown in [55] that $|\boldsymbol{u}|_\epsilon$ is a good approximation for $|\boldsymbol{u}|$ and the partial derivatives of $|\boldsymbol{u}|_\epsilon$ approximate the partial derivatives of $|\boldsymbol{u}|$ very well. Particularly, $|\boldsymbol{u}|_\epsilon - |\boldsymbol{u}| < \epsilon$. Now consider the function $\boldsymbol{f}_\epsilon : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by

$$\boldsymbol{f}_\epsilon(\boldsymbol{u}) = |\boldsymbol{u}|_\epsilon \boldsymbol{u} \ ,$$

$$\boldsymbol{f}_\epsilon(\boldsymbol{u}) = \begin{bmatrix} \boldsymbol{f}_{\epsilon,1}(\boldsymbol{u}) \\ \\ \boldsymbol{f}_{\epsilon,2}(\boldsymbol{u}) \end{bmatrix} = \begin{bmatrix} |\boldsymbol{u}|_\epsilon u_1 \\ \\ |\boldsymbol{u}|_\epsilon u_2 \end{bmatrix} \ .$$

The Taylor series representation of $\boldsymbol{f}_\epsilon(\boldsymbol{u})$ about $\boldsymbol{u}_H \in \mathbb{R}^2$ is given by,

$$\boldsymbol{f}_\epsilon(\boldsymbol{u}) = \boldsymbol{f}_\epsilon(\boldsymbol{u}_H) + \mathcal{D}\boldsymbol{f}_\epsilon(\boldsymbol{u}_H)(\boldsymbol{u} - \boldsymbol{u}_H) + \frac{1}{2!}\mathcal{D}^2\boldsymbol{f}_\epsilon(\boldsymbol{\zeta}_H)(\boldsymbol{u} - \boldsymbol{u}_H)^2 \ , \tag{4.5}$$

where $\boldsymbol{\zeta}_H = t\ \boldsymbol{u} + (1-t)\ \boldsymbol{u}_H$, for some t in $[0,1]$ and $(\boldsymbol{u} - \boldsymbol{u}_H)^2$ is expressed as Kronecker product $(\boldsymbol{u} - \boldsymbol{u}_H) \otimes (\boldsymbol{u} - \boldsymbol{u}_H)$. Precisely, $(\boldsymbol{u} - \boldsymbol{u}_H)^2$ is equal to

$$[(u_1 - u_{H,1})^2, (u_1 - u_{H,1})(u_2 - u_{H,2}), (u_2 - u_{H,2})(u_1 - u_{H,1}), (u_2 - u_{H,2})^2]^T.$$

The matrix of the first and second derivatives are as follows:

$\mathcal{D}\boldsymbol{f}_\epsilon(\boldsymbol{u}_H)$ and $\mathcal{D}^2\boldsymbol{f}_\epsilon(\boldsymbol{\zeta}_H)$ are given as follows,

$$\mathcal{D}\boldsymbol{f}_\epsilon(\boldsymbol{u}_H) = \begin{pmatrix} \frac{\partial \boldsymbol{f}_{\epsilon,1}(\boldsymbol{u}_H)}{\partial u_1} & \frac{\partial \boldsymbol{f}_{\epsilon,1}(\boldsymbol{u}_H)}{\partial u_2} \\ \frac{\partial \boldsymbol{f}_{\epsilon,2}(\boldsymbol{u}_H)}{\partial u_1} & \frac{\partial \boldsymbol{f}_{\epsilon,2}(\boldsymbol{u}_H)}{\partial u_2} \end{pmatrix} \ , \tag{4.6}$$

$$\mathcal{D}^2\boldsymbol{f}_\epsilon(\boldsymbol{\zeta}_H) = \begin{pmatrix} \frac{\partial^2 \boldsymbol{f}_{\epsilon,1}(\boldsymbol{\zeta}_H)}{\partial u_1^2} & \frac{\partial^2 \boldsymbol{f}_{\epsilon,1}(\boldsymbol{\zeta}_H)}{\partial u_1 \partial u_2} & \frac{\partial^2 \boldsymbol{f}_{\epsilon,1}(\boldsymbol{\zeta}_H)}{\partial u_2 \partial u_1} & \frac{\partial^2 \boldsymbol{f}_{\epsilon,1}(\boldsymbol{\zeta}_H)}{\partial u_2^2} \\ \frac{\partial^2 \boldsymbol{f}_{\epsilon,2}(\boldsymbol{\zeta}_H)}{\partial u_1^2} & \frac{\partial^2 \boldsymbol{f}_{\epsilon,2}(\boldsymbol{\zeta}_H)}{\partial u_1 \partial u_2} & \frac{\partial^2 \boldsymbol{f}_{\epsilon,2}(\boldsymbol{\zeta}_H)}{\partial u_2 \partial u_1} & \frac{\partial^2 \boldsymbol{f}_{\epsilon,2}(\boldsymbol{\zeta}_H)}{\partial u_2^2} \end{pmatrix} \ . \tag{4.7}$$

In view of (4.5), for any $\boldsymbol{\varphi} = (\varphi_1, \varphi_2) \in \mathbb{R}^2$,

$$\begin{aligned}\boldsymbol{f}_\epsilon(\boldsymbol{u}) \cdot \boldsymbol{\varphi} = \sum_{i=1}^{2} \boldsymbol{f}_{\epsilon,i}(\boldsymbol{u}_H)\varphi_i + \sum_{i=1}^{2}\sum_{j=1}^{2} \frac{\partial \boldsymbol{f}_{\epsilon,i}(\boldsymbol{u}_H)}{\partial u_j}(u_j - u_{H,j})\varphi_i \\ + \sum_{k=1}^{2}\sum_{j=1}^{2}\sum_{i=1}^{2} \frac{\partial^2 \boldsymbol{f}_{\epsilon,k}(\boldsymbol{\zeta}_H)}{\partial u_i \partial u_j}(u_i - u_{H,i})(u_j - u_{H,j})\varphi_k \ . \end{aligned} \tag{4.8}$$

We now present the two-level algorithm.

4.2.1 Two-level algorithm

Define the following finite dimensional subspaces.

$$\begin{aligned} X_H &= \left\{ \boldsymbol{v} \in [L^2(\Omega)]^2; \boldsymbol{v}|_T T \in \mathbb{P}_0 \text{ for all } T \in \mathcal{T}_h, \right\}, \\ Q_H &= \left\{ q \in C^0(\bar{\Omega}); q|_T \in \mathbb{P}_1 \text{ for all } T \in \mathcal{T}_H \right\}, \\ M_H &= Q_H \cap L_0^2(\Omega). \end{aligned} \tag{4.9}$$

Let $X_H, X_h \subset X, M_H, M_h \subset M$ denote finite element spaces as defined in (4.9) and (3.6) with $X_H \subset X_h, M_H \subset M_h$, $H \gg h$.

Step 1: Solve the nonlinear system on a coarse mesh $\mathcal{T}_H$ with meshsize H: For all $(\boldsymbol{\varphi}_H, q_H) \in X_H \times M_H$, find $(\boldsymbol{u}_H, p_H) \in X_H \times M_H$ such that,

$$\frac{\mu}{\rho}\int_\Omega (K^{-1}\boldsymbol{u}_H) \cdot \boldsymbol{\varphi}_H d\boldsymbol{x} + \frac{\beta}{\rho}\int_\Omega |\boldsymbol{u_H}|\boldsymbol{u}_H \cdot \boldsymbol{\varphi}_H d\boldsymbol{x} + \int_\Omega \bigtriangledown p_H \cdot \boldsymbol{\varphi}_H d\boldsymbol{x} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi}_H d\boldsymbol{x}, \tag{4.10}$$

$$\int_\Omega \bigtriangledown q_H \cdot \boldsymbol{u_H} d\boldsymbol{x} = -\int_\Omega b q_H \boldsymbol{d}x + \int_\Gamma g q_H \boldsymbol{d}\sigma, \tag{4.11}$$

Step 2: Solve the linear system on a fine mesh $\mathcal{T}_h$ with meshsize h:

For all $(\boldsymbol{\varphi}_h, q_h) \in X_h \times M_h$, find $(\boldsymbol{u}_h, p_h) \in X_h \times M_h$ such that,

$$\frac{\mu}{\rho}\int_\Omega (K^{-1}\boldsymbol{u}_h)\cdot\boldsymbol{\varphi}_h dx + \frac{\beta}{\rho}\int_\Omega \mathcal{D}\boldsymbol{f}_\epsilon(\boldsymbol{u}_H)\boldsymbol{u}_h\cdot\boldsymbol{\varphi}_h d\boldsymbol{x} + \int_\Omega \bigtriangledown p_h\cdot\boldsymbol{\varphi}_h d\boldsymbol{x}$$
$$= \int_\Omega \boldsymbol{f}\boldsymbol{\varphi}_h dx - \frac{\beta}{\rho}\int_\Omega |\boldsymbol{u_H}|_{\boldsymbol{\epsilon}}\boldsymbol{u_H}\cdot\boldsymbol{\varphi}_h d\boldsymbol{x} + \frac{\beta}{\rho}\int_\Omega \mathcal{D}\boldsymbol{f}_\epsilon(\boldsymbol{u}_H)\boldsymbol{u}_H\cdot\boldsymbol{\varphi}_h d\boldsymbol{x}, \tag{4.12}$$

and

$$\int_\Omega \bigtriangledown q_h\cdot\boldsymbol{u}_h d\boldsymbol{x} = -\int_\Omega bq_h\boldsymbol{d}x + \int_\Gamma gq_h\boldsymbol{d}\sigma. \tag{4.13}$$

Lemma 4.1 *Suppose* $(\boldsymbol{u}_H, p_H)$ *is the solution of* (4.10)-(4.11). *Then the first and second order partial derivatives of* $\boldsymbol{f}_\epsilon(\boldsymbol{u}_H)$ *exist, they are continuous and bounded.*

Proof. Direct differentiation shows that

$$\frac{\partial\boldsymbol{f}_{\epsilon,i}(\boldsymbol{u}_H)}{\partial u_{H_j}} = (u_{H_i}u_{H_j} + |\boldsymbol{u}_H|_\epsilon^2\delta_{ij})|\boldsymbol{u}_H|_\epsilon^{-1} \quad i,j=1,2 \text{ ,where } \delta_{\text{ij}} = \begin{cases} 1, & i\ = j \\ 0, & i \neq j \ . \end{cases} \tag{4.14}$$

There exists a constant C_1 such that

$$\left|\frac{\partial\boldsymbol{f}_{\epsilon,i}(\boldsymbol{u}_H)}{\partial u_j}\right| \le (C_1 + \epsilon) \tag{4.15}$$

$$\frac{\partial^2\boldsymbol{f}_{\epsilon,k}(\boldsymbol{u}_H)}{\partial u_{H_i}u_{H_j}} = (2u_{H_i}|\boldsymbol{u}_H|_\epsilon^2\delta_{ijk} - u_{H_i}u_{H_j}u_{H_k} + u_s|\boldsymbol{u}_H|_\epsilon^2)|\boldsymbol{u}_H|_\epsilon^{-3} \quad i,j,k=1,2,$$

$$s = \begin{cases} k, & i = j = k \\ k, & i = j \neq k \\ j, & i \neq j, k = i \\ i, & i \neq j, k = j \quad . \end{cases}$$

There exists a constant C_2 such that

$$\left| \frac{\partial^2 \boldsymbol{f}_{\epsilon,k}(\boldsymbol{u}_H)}{\partial u_{H_i} u_{H_j}} \right| \leq C_2 \quad i, j, k = 1, 2. \tag{4.16}$$

The bounds in (4.15) and (4.16) use the fact that each component of $\boldsymbol{u}_H$ is piecewise constant. ∎

Lemma 4.2 *The matrix $\mathcal{D}\boldsymbol{f}_\epsilon(\boldsymbol{u})$ is a symmetric and positive definite.*

Proof. We deduce from (4.14) and (4.6) that

$$\mathcal{D}\boldsymbol{f}_\epsilon(\boldsymbol{u}) = \begin{pmatrix} \frac{u_1^2 + |\boldsymbol{u}|_\epsilon^2}{|\boldsymbol{u}|_\epsilon} & \frac{u_1 u_2}{|\boldsymbol{u}|_\epsilon} \\ \frac{u_1 u_2}{|\boldsymbol{u}|_\epsilon} & \frac{u_2^2 + |\boldsymbol{u}|_\epsilon^2}{|\boldsymbol{u}|_\epsilon} \end{pmatrix} ,$$

Observe that

$$\begin{pmatrix} \frac{u_1^2 + |\boldsymbol{u}|_\epsilon^2}{|\boldsymbol{u}|_\epsilon} & \frac{u_1 u_2}{|\boldsymbol{u}|_\epsilon} \\ \frac{u_1 u_2}{|\boldsymbol{u}|_\epsilon} & \frac{u_2^2 + |\boldsymbol{u}|_\epsilon^2}{|\boldsymbol{u}|_\epsilon} \end{pmatrix} = \frac{1}{|\boldsymbol{u}|_\epsilon} \left[\begin{pmatrix} u_1^2 & u_1 u_2 \\ u_1 u_2 & u_2^2 \end{pmatrix} + |\boldsymbol{u}|_\epsilon^2 \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \right] .$$

Next,

$$\begin{vmatrix} u_1^2 - \lambda & u_1 u_2 \\ u_1 u_2 & u_2^2 - \lambda \end{vmatrix} = -\lambda(u_1^2 + u_2^2 + \lambda) = 0,$$

so $\lambda = 0$ or $\lambda = -(u_1^2 + u_2^2)$. It follows that the eigenvalues of $\mathcal{D}\boldsymbol{f}_\epsilon(\boldsymbol{u})$ are either $|\boldsymbol{u}|_\epsilon^2$ or ϵ^2. ∎

The well-posedness of the two-level algorithm will be presented in the next section.

4.2.2 Well-posedness of the discrete problem in step 2

The well-posedness of (4.10)-(4.11) is a particular case (m=2) of the generalized case discussed in chapter 3. See also [31]. We now discuss the well-posedness of (4.12)-(4.13).

Existence and uniqueness

Let V_h represent the discrete analogue of V as defined in (3.11). Define the bilinear forms

$$\bar{a} : X_h \times X_h \longrightarrow \mathbb{R}, \quad \bar{b} : M_h \times X_h \longrightarrow \mathbb{R}, \text{ by}$$

$$\bar{a}(\boldsymbol{u}_h, \boldsymbol{\varphi_h}) = \frac{\mu}{\rho} \int_\Omega (K^{-1}\boldsymbol{u}_h) \cdot \boldsymbol{\varphi}_h dx + \frac{\beta}{\rho} \int_\Omega \mathcal{D}\boldsymbol{f}_\epsilon(\boldsymbol{u}_H)\boldsymbol{u}_h \cdot \boldsymbol{\varphi}_h d\boldsymbol{x} \ , \tag{4.17}$$

$$\bar{b}(q_h, \boldsymbol{u_h}) = \int_\Omega \bigtriangledown q_h \boldsymbol{u}_h d\boldsymbol{x} \ . \tag{4.18}$$

and these linear functionals

$$\langle \boldsymbol{f}, \boldsymbol{\varphi}_h \rangle_{X^* \times X_h} = \int_\Omega \boldsymbol{f} \boldsymbol{\varphi}_h dx - \frac{\beta}{\rho} \int_\Omega |\boldsymbol{u_H}|_\epsilon \boldsymbol{u_H} \cdot \boldsymbol{\varphi}_h d\boldsymbol{x}$$
$$+ \frac{\beta}{\rho} \int_\Omega \mathcal{D} \boldsymbol{f}_\epsilon(\boldsymbol{u}_H) \boldsymbol{u}_H \cdot \boldsymbol{\varphi}_h d\boldsymbol{x} \tag{4.19}$$
$$\langle g, q \rangle_{M^* \times M_h} = - \int_\Omega b q_h \boldsymbol{d}x + \int_\Gamma g q_h \boldsymbol{d}\sigma. \tag{4.20}$$

Then, Problem (4.12)-(4.13) can be written as follows: Given $(\boldsymbol{f}, g) \in X^* \times M^*$, find a pair of function $(\boldsymbol{u}_h, p_h) \in X_h \times M_h$ such that

$$\begin{cases} \bar{a}(\boldsymbol{u}_h, \boldsymbol{\varphi}_h) + \bar{b}(p_h, \boldsymbol{\varphi}_h) = \langle \boldsymbol{f}, \boldsymbol{\varphi}_h \rangle_{X^* \times X}, \quad \forall \boldsymbol{\varphi}_h \in X_h, \\ \bar{b}(\boldsymbol{u}_h, q_h) = \langle g, q \rangle_{M^* \times M_h}, \quad \forall q_h \in M_h, \end{cases} \tag{4.21}$$

Then, Problem (4.21) reduces to: Find $\boldsymbol{u}_h \in V_h$ such that

$$\bar{a}(\boldsymbol{u_h}, \boldsymbol{\varphi_h}) = \langle \boldsymbol{f}, \boldsymbol{\varphi}_h \rangle_{X^* \times X_h}, \ \forall \boldsymbol{\varphi}_h \in X_h \tag{4.22}$$

Lemma 4.3 *The bilinear form $\bar{a}$ is coercive on V_h. That is, there exists $\alpha > 0$ such that*

$$\bar{a}(\boldsymbol{u}_h, \boldsymbol{u}_h) \geq \alpha \|\boldsymbol{u}_h\|^2_{[L^2(\Omega)]^d}, \ \forall \boldsymbol{u}_h \in V_h, \tag{4.23}$$

and the bilinear form $\bar{b}$ satisfies

$$\forall q_h \in M_h \inf_{q \in M_h} \sup_{\boldsymbol{u}_h \in X_h} \frac{\bar{b}(q_h, \boldsymbol{u}_h)}{\|\boldsymbol{u}_h\|_{[L^3(\Omega)]^d} \| \bigtriangledown q_h\|_{[L^{3/2}(\Omega)]^d}} = 1. \tag{4.24}$$

Proof.

$$\bar{a}(\boldsymbol{u_h}, \boldsymbol{u_h}) = \frac{\mu}{\rho}\int_\Omega (K^{-1}\boldsymbol{u}_h)\cdot \boldsymbol{u_h} dx + \frac{\beta}{\rho}\int_\Omega \mathcal{D}\boldsymbol{f}_\epsilon(\boldsymbol{u}_H)\boldsymbol{u}_h \cdot \boldsymbol{u_h} d\boldsymbol{x}. \tag{4.25}$$

Although,

$$\begin{aligned}
\frac{\beta}{\rho}\int_\Omega \mathcal{D}\boldsymbol{f}_\epsilon(\boldsymbol{u}_H)\boldsymbol{u}_h \cdot \boldsymbol{u_h} d\boldsymbol{x} &= \frac{\beta}{\rho}\int_\Omega \boldsymbol{u}_h^\top \mathcal{D}\boldsymbol{f}_\epsilon(\boldsymbol{u}_H)\boldsymbol{u_h} d\boldsymbol{x}, \\
&= \frac{\beta}{\rho}\int_\Omega \left([\mathcal{D}\boldsymbol{f}_\epsilon(\boldsymbol{u}_H)]^{\frac{1}{2}}\boldsymbol{u}_h\right)^\top \left([\mathcal{D}\boldsymbol{f}_\epsilon(\boldsymbol{u}_H)]^{\frac{1}{2}}\boldsymbol{u}_h\right), \\
&= \frac{\beta}{\rho}\|[\mathcal{D}\boldsymbol{f}_\epsilon(\boldsymbol{u}_H)]^{\frac{1}{2}}\|^2 \geq 0.
\end{aligned} \tag{4.26}$$

So, (4.25) becomes

$$\bar{a}(\boldsymbol{u_h}, \boldsymbol{u_h}) \geq \frac{\mu}{\rho}\lambda_s \|\boldsymbol{u_h}\|^2_{[L^2(\Omega)]^d}\ , \quad \forall \boldsymbol{u_h} \in V_h. \tag{4.27}$$

Hence, there exists a unique $\boldsymbol{u}_h \in X_h$ that satisfies (4.22). In view of the inf-sup condition (4.24) (see [31, Proposition 6] for proof) and Theorem 1.6, we conclude that there exists a unique $p_h \in M_h$ such that $(\boldsymbol{u}_h, p_h)$ solves (4.12)-(4.13). ∎

Stability

The stability of **Step 1** of the two level algorithm was presented in the following theorem.

Theorem 4.1 *[31, Theorem 3] For any data* $(b, g) \in L^{\frac{6}{5}}(\Omega) \times L^{\frac{3}{2}}(\Gamma)$ *satisfying* (1.4), $\boldsymbol{u}_{H_\ell} \in X_H$ *satisfying* (4.11). *The unique solution* $(\boldsymbol{u}_H, p_H) \in X_H \times M_H$ *of*

(4.10)-(4.11) *satisfies the following bounds,*

$$\|\boldsymbol{u}_H\|_{[L^3(\Omega)]^d} \le \left(\|\boldsymbol{u}_{H_\ell}\|_{[L^3(\Omega)]^d} + \frac{3}{4}\frac{\mu}{\rho}\frac{\|K^{-1}\|_{L^\infty(\Omega)}}{\lambda_s}\|\boldsymbol{u}_{H_\ell}\|^2_{[L^2(\Omega)]^d} \right)^{\frac{1}{3}}, \tag{4.28}$$

and

$$\| \bigtriangledown p_H\|_{[L^{\frac{3}{2}}(\Omega)]^d} \le \frac{\mu}{\rho}\|K^-\|_{L^\infty(\Omega)}\|\boldsymbol{u}\|_{[L^{\frac{3}{2}}(\Omega)]^d} + \frac{\beta}{\rho}\|\boldsymbol{u}\|^2_{[L^3(\Omega)]^d} \tag{4.29}$$

where $\boldsymbol{u}_{H_\ell}$ *satisfies:*

$$\|\boldsymbol{u}_{H_\ell}\|_{[L^3(\Omega))]^d} \le C\left(\|b\|_{L^{\frac{3d}{3+d}}(\Omega)} + \|g\|_{L^{\frac{3(d-1)}{d}}(\Gamma)} \right). \tag{4.30}$$

where C *is a constant depending on* Ω *only.*

Now we present the stability of **Step 2**.

Theorem 4.2 *Let* $(\boldsymbol{u_H}, p_H) \in X_H \times M_H$ *be the unique solution of* $(4.10) - (4.11)$, $(\boldsymbol{u_h}, p_h) \in X_h \times M_h$ *be the unique solution of* $(4.12) - (4.13)$, *then for each* $(b, g) \in L^{\frac{6}{5}}(\Omega) \times L^{\frac{3}{2}}(\Gamma)$ *satisfying* (1.4). *The following bounds holds:*

$$\|u_h\|_{[L^2(\Omega)]^d} \le \frac{\rho}{\mu\lambda_s}\|\boldsymbol{f}\|_{[L^2(\Omega)]^d} + \frac{\beta}{\mu\lambda_s}(C+\epsilon)\|\mathcal{D}\boldsymbol{f}_\epsilon(\boldsymbol{u}_H)\|_{L^\infty(\Omega)}\|\boldsymbol{u}_H\|_{[L^2(\Omega)]^d} \tag{4.31}$$

and

$$\begin{aligned} \| \bigtriangledown p_h\|_{[L^{\frac{3}{2}}(\Omega)]^d} \le \|\boldsymbol{f}\|_{[L^{\frac{3}{2}}(\Omega)]^d} &+ \left(\frac{\mu}{\rho}\|K^{-1}\|_{L^\infty(\Omega)} + \frac{\beta}{\rho}(C_1+\epsilon)\right)\|\boldsymbol{u}_h\|_{[L^{\frac{3}{2}}(\Omega)]^d} \\ &+ \|\boldsymbol{u}_H\|_{[L^{\frac{3}{2}}(\Omega)]^d}, \end{aligned} \tag{4.32}$$

where C is a positive constant.

Proof. Applying (4.27) in (4.22) gives,

$$\frac{\mu\lambda_s}{\rho}\|\boldsymbol{u}_h\|^2_{[L^2(\Omega)]^d} \leq \|\boldsymbol{f}\|_{[L^2(\Omega)]^d}\|\boldsymbol{u}_h\|_{[L^2(\Omega)]^d} + \frac{\beta}{\rho}(C+\epsilon)\|\boldsymbol{u}_H\|_{[L^2(\Omega)]^d}\|\boldsymbol{u}_h\|_{[L^2(\Omega)]^d} + \frac{\beta}{\rho}\|\mathcal{D}\boldsymbol{f}_\epsilon(\boldsymbol{u}_H)\|_{L^\infty(\Omega)}\|\boldsymbol{u}_H\|_{[L^2(\Omega)]^d}\|\boldsymbol{u}_h\|_{[L^2(\Omega)]^d}.$$

Hence (4.31) follows from the last inequality above. Now define,

$$\tilde{\mathcal{A}}(\boldsymbol{u}_h) = \frac{\mu}{\rho}K^{-1}\boldsymbol{u}_h + \frac{\beta}{\rho}\mathcal{D}\boldsymbol{f}_\epsilon(\boldsymbol{u}_H)\boldsymbol{u}_h - (\boldsymbol{f} - \frac{\beta}{\rho}|\boldsymbol{u_H}|_{\boldsymbol{\epsilon}}\boldsymbol{u_H} + \frac{\beta}{\rho}\mathcal{D}\boldsymbol{f}_\epsilon(\boldsymbol{u}_H)\boldsymbol{u}_H).$$

Proposition 4.1 *The operator $\tilde{\mathcal{A}}$ satisfies the following bound:*

$$\|\tilde{\mathcal{A}}(\boldsymbol{u}_h)\|_{[L^{\frac{3}{2}}(\Omega)]^d} \leq \left(\frac{\mu}{\rho}\|K^{-1}\|_{L^\infty(\Omega)} + \frac{\beta}{\rho}(C+\epsilon)\right)\|\boldsymbol{u}_h\|_{[L^{\frac{3}{2}}(\Omega)]^d} + \|\boldsymbol{f}\|_{[L^{\frac{3}{2}}(\Omega)]^d} + \frac{\beta}{\rho}\|\mathcal{D}\boldsymbol{f}_\epsilon(\boldsymbol{u}_H)\|_{L^\infty(\Omega)}\|\boldsymbol{u}_H\|_{[L^{\frac{3}{2}}(\Omega)]^d}. \tag{4.33}$$

Proof. We estimate as follows:

$$\left|\langle\tilde{\mathcal{A}}(\boldsymbol{u}_h), \boldsymbol{\varphi}_h\rangle_{[L^{\frac{3}{2}}(\Omega)]^d\times[L^3(\Omega)]^d}\right| \leq \frac{\mu}{\rho}\|K^{-1}\|_{L^\infty(\Omega)}\|\boldsymbol{u}_h\|_{[L^{\frac{3}{2}}(\Omega)]^d}\|\boldsymbol{\varphi}_h\|_{[L^3(\Omega)]^d} + \frac{\beta}{\rho}(C+\epsilon)\|\boldsymbol{u}_h\|_{[L^{\frac{3}{2}}(\Omega)]^d}\|\boldsymbol{\varphi}_h\|_{[L^3(\Omega)]^d} + \|\boldsymbol{f}\|_{[L^{\frac{3}{2}}(\Omega)]^d}\|\boldsymbol{\varphi}_h\|_{[L^3(\Omega)]^d} + \frac{\beta}{\rho}\|\mathcal{D}\boldsymbol{f}_\epsilon(\boldsymbol{u}_H)\|_{L^\infty(\Omega)}\|\boldsymbol{u}_H\|_{[L^{\frac{3}{2}}(\Omega)]^d}\|\boldsymbol{\varphi}_h\|_{[L^3(\Omega)]^d}. \tag{4.34}$$

The bound in (4.33) is deduced from (4.34). ▮

Equation (4.22) implies that $\tilde{\mathcal{A}}(\boldsymbol{u}_h)$ belongs to $(V_h^{\perp})$. Thanks to the infsup condition in Lemma (4.3) and finite dimensional variant of Babuska-Brezzi's theory [5], there exists a unique $p_h \in M_h$ such that,

$$\bigtriangledown p_h = \tilde{\mathcal{A}}(\boldsymbol{u}_h). \tag{4.35}$$

Thanks to Estimate (4.33), (4.32) follows from (4.35). ▮

4.3 Error estimates

The error estimates for (4.10)-(4.11) have been established in [31] and summarized in the following theorem.

Theorem 4.3 *[31, 57] Let* $(\boldsymbol{u_H}, p_H)$ *be solution of* (4.10)-(4.11)*,* $(\boldsymbol{u}, p)$ *the solution of* (2.1)-(2.2) *and assume* $\boldsymbol{u}$ *belongs to* $W^{1,4}(\Omega)^d$*. Then we have the following error bounds.*

$$\|u - u_H\|_{[L^2(\Omega)]^d} \leq CH|\boldsymbol{u}|_{W^{1,4}(\Omega)}. \tag{4.36}$$

In addition, suppose that the solution p *of* $(2.1) - (2.2)$ *belongs to* $W^{2,\frac{3}{2}}$*, then*

$$\| \bigtriangledown (p - p_H)\|_{[L^{\frac{3}{2}}(\Omega)]^d} \leq CH\left(|p|_{W^{2,\frac{3}{2}}} + \|\boldsymbol{u}\|_{W^{1,4}(\Omega)}\right). \tag{4.37}$$

To obtain the error estimates for the two- level algorithm, we start by rewriting (2.1)-(2.2) as: find $(\boldsymbol{u}, p) \in X \times M$ such that,

$$
\begin{aligned}
&\frac{\mu}{\rho}\int_{\Omega}(K^{-1}\boldsymbol{u})\cdot\boldsymbol{\varphi}dx + \frac{\beta}{\rho}\int_{\Omega}|\boldsymbol{u_H}|_{\boldsymbol{\epsilon}}\boldsymbol{u}_H\cdot\boldsymbol{\varphi}d\boldsymbol{x} + \frac{\beta}{\rho}\int_{\Omega}\mathcal{D}\boldsymbol{f}_{\epsilon}(\boldsymbol{u}_H)(\boldsymbol{u}-\boldsymbol{u}_H)\cdot\boldsymbol{\varphi}d\boldsymbol{x} + \int_{\Omega}\bigtriangledown p\cdot\boldsymbol{\varphi}d\boldsymbol{x} \\
&= \int_{\Omega}\boldsymbol{f}\boldsymbol{\varphi}dx - \frac{\beta}{\rho}\int_{\Omega}|\boldsymbol{u}|\boldsymbol{u}\cdot\boldsymbol{\varphi}d\boldsymbol{x} + \frac{\beta}{\rho}\int_{\Omega}|\boldsymbol{u_H}|_{\boldsymbol{\epsilon}}\boldsymbol{u}_H\cdot\boldsymbol{\varphi}d\boldsymbol{x} \\
&\quad + \frac{\beta}{\rho}\int_{\Omega}\mathcal{D}\boldsymbol{f}_{\epsilon}(\boldsymbol{u}_H)(\boldsymbol{u}-\boldsymbol{u}_H)\cdot\boldsymbol{\varphi}d\boldsymbol{x},\ \forall\boldsymbol{\varphi}\in X, \qquad (4.38)
\end{aligned}
$$

and

$$
\int_{\Omega}\bigtriangledown q\cdot\boldsymbol{u}d\boldsymbol{x} = -\int_{\Omega}bq\boldsymbol{d}x + \int_{\Gamma}gq\boldsymbol{d}\sigma,\ \forall q\in M\ . \qquad (4.39)
$$

In view of (4.5), (4.38)-(4.39) becomes,

$$
\begin{aligned}
&\frac{\mu}{\rho}\int_{\Omega}(K^{-1}\boldsymbol{u})\cdot\boldsymbol{\varphi}dx + \frac{\beta}{\rho}\int_{\Omega}|\boldsymbol{u_H}|_{\boldsymbol{\epsilon}}\boldsymbol{u}_H\cdot\boldsymbol{\varphi}d\boldsymbol{x} + \frac{\beta}{\rho}\int_{\Omega}\mathcal{D}\boldsymbol{f}_{\epsilon}(\boldsymbol{u}_H)(\boldsymbol{u}-\boldsymbol{u}_H)\cdot\boldsymbol{\varphi}d\boldsymbol{x} + \int_{\Omega}\bigtriangledown p\cdot\boldsymbol{\varphi}d\boldsymbol{x} \\
&= \int_{\Omega}\boldsymbol{f}\boldsymbol{\varphi}dx - \frac{\beta}{\rho}\int_{\Omega}|\boldsymbol{u}|\boldsymbol{u}\cdot\boldsymbol{\varphi}d\boldsymbol{x} + \frac{\beta}{\rho}\int_{\Omega}|\boldsymbol{u}|_{\boldsymbol{\epsilon}}\boldsymbol{u}\cdot\boldsymbol{\varphi}d\boldsymbol{x} \\
&\quad - \frac{\beta}{2!\rho}\int_{\Omega}\mathcal{D}^2\boldsymbol{f}_{\epsilon}(\boldsymbol{\zeta}_H)(\boldsymbol{u}-\boldsymbol{u}_H)^2\cdot\boldsymbol{\varphi}d\boldsymbol{x},\ \forall\boldsymbol{\varphi}\in X, \qquad (4.40)
\end{aligned}
$$

and

$$
\int_{\Omega}\bigtriangledown q\cdot\boldsymbol{u}d\boldsymbol{x} = -\int_{\Omega}bq\boldsymbol{d}x + \int_{\Gamma}gq\boldsymbol{d}\sigma,\ \forall q\in M\ . \qquad (4.41)
$$

Furthermore, (4.12)-(4.13) can be re-arranged as follows:

for all $(\boldsymbol{\varphi_h}, q_h) \in X_h \times M_h$, find $(\boldsymbol{u}_h, p_h) \in X_h \times M_h$ such that,

$$\frac{\mu}{\rho}\int_\Omega (K^{-1}\boldsymbol{u}_h)\cdot\boldsymbol{\varphi}_h dx + \frac{\beta}{\rho}\int_\Omega |\boldsymbol{u_H}|_{\boldsymbol{\epsilon}}\boldsymbol{u_H}\cdot\boldsymbol{\varphi}_h d\boldsymbol{x} + \frac{\beta}{\rho}\int_\Omega \mathcal{D}\boldsymbol{f}_\epsilon(\boldsymbol{u}_H)(\boldsymbol{u}_h - \boldsymbol{u}_H)\cdot\boldsymbol{\varphi}_h d\boldsymbol{x}$$
$$+ \int_\Omega \bigtriangledown p_h \cdot \boldsymbol{\varphi}_h d\boldsymbol{x} = \int_\Omega \boldsymbol{f}\boldsymbol{\varphi}_h dx, \tag{4.42}$$

and

$$\int_\Omega \bigtriangledown q_h \cdot \boldsymbol{u}_h d\boldsymbol{x} = -\int_\Omega b q_h \boldsymbol{d}x + \int_\Gamma g q_h \boldsymbol{d\sigma}. \tag{4.43}$$

4.3.1 Error estimates for velocity

The error estimate for the velocity is given by the following Proposition.

Proposition 4.2 *Let $(\boldsymbol{u}, p)$ be solution of (2.1)-(2.2), $(\boldsymbol{u_H}, p_H)$ be solution of (4.10)-(4.11) and $(\boldsymbol{u_h}, p_h)$ be the solution of (4.12)-(4.13). Suppose $(\boldsymbol{u}, p) \in L^3(\Omega)\times W^{1,2}(\Omega)$, then for any $\boldsymbol{w_h} \in X_h$ satisfying the constraint in (4.13), the following error estimate holds:*

$$\|\boldsymbol{u}-\boldsymbol{u_h}\|_{[L^2(\Omega)]^d} \leq \left(1 + \frac{\|K^{-1}\|_{L^\infty(\Omega)}}{\lambda_s} + \frac{\beta}{\mu\lambda_s}(C_1+\epsilon)\right)\|\boldsymbol{u}-\boldsymbol{w_h}\|_{[L^2(\Omega)]^d}$$
$$+ \frac{\beta C}{2!\mu\lambda_s}\|\boldsymbol{u}-\boldsymbol{u}_H\|^2_{[L^2(\Omega)]^d} + \frac{\rho}{\mu\lambda_s}\|\bigtriangledown(p - I_h(p))\|_{[L^2(\Omega)]^d}$$
$$+ \frac{\beta}{\rho}\epsilon\|\boldsymbol{u}\|_{[L^2(\Omega)]^d}\ , \tag{4.44}$$

where C and C_1 are positive constants independent of h and H.

Proof. Subtracting (4.42) from (4.40), we get,

$$
\begin{aligned}
&\frac{\mu}{\rho}\int_{\Omega} K^{-1}\boldsymbol{(u-u_h)}\cdot\boldsymbol{\varphi_h} d\boldsymbol{x}+\frac{\beta}{\rho}\int_{\Omega}\mathcal{D}\boldsymbol{f}_{\epsilon}(\boldsymbol{u}_H)(\boldsymbol{u}-\boldsymbol{u}_h)\cdot\boldsymbol{\varphi}_h d\boldsymbol{x}+\int_{\Omega}\bigtriangledown(p-p_h)\cdot\boldsymbol{\varphi}_h d\boldsymbol{x}\\
&=\frac{\beta}{2!\rho}\int_{\Omega}\mathcal{D}^2\boldsymbol{f}_{\epsilon}(\boldsymbol{\zeta}_H)\cdot(\boldsymbol{u}-\boldsymbol{u}_H)^2\cdot\boldsymbol{\varphi}_h d\boldsymbol{x}+\frac{\beta}{\rho}\int_{\Omega}(|\boldsymbol{u}|_{\epsilon}-|\boldsymbol{u}|)\boldsymbol{u}\cdot\boldsymbol{\varphi} d\boldsymbol{x},\quad\forall\boldsymbol{\varphi}_h\in V_h.
\end{aligned}
$$

Inserting $I_h(p)$ and using the fact that $\boldsymbol{\varphi}_h \in V_h$ results to

$$
\begin{aligned}
&\frac{\mu}{\rho}\int_{\Omega} K^{-1}\boldsymbol{(u-u_h)}\cdot\boldsymbol{\varphi_h} d\boldsymbol{x}+\frac{\beta}{\rho}\int_{\Omega}\mathcal{D}\boldsymbol{f}_{\epsilon}(\boldsymbol{u}_H)(\boldsymbol{u}-\boldsymbol{u}_h)\cdot\boldsymbol{\varphi}_h d\boldsymbol{x}\\
&=\frac{\beta}{2!\rho}\int_{\Omega}\mathcal{D}^2\boldsymbol{f}_{\epsilon}(\zeta_H)\cdot(\boldsymbol{u}-\boldsymbol{u}_H)^2\cdot\boldsymbol{\varphi}_h d\boldsymbol{x}+\int_{\Omega}\bigtriangledown(I_h(p)-p)\cdot\boldsymbol{\varphi_h} d\boldsymbol{x}\\
&\quad+\frac{\beta}{\rho}\int_{\Omega}(|\boldsymbol{u}|_{\epsilon}-|\boldsymbol{u}|)\boldsymbol{u}\cdot\boldsymbol{\varphi} d\boldsymbol{x},\quad\forall\boldsymbol{\varphi_h}\in V_h\ .
\end{aligned}
\tag{4.45}
$$

Hence

$$
\begin{aligned}
&\frac{\mu}{\rho}\int_{\Omega} K^{-1}(\boldsymbol{u}_h-\boldsymbol{w}_h)\cdot\boldsymbol{\varphi_h} d\boldsymbol{x}+\frac{\beta}{\rho}\int_{\Omega}\mathcal{D}\boldsymbol{f}_{\epsilon}(\boldsymbol{u}_H)(\boldsymbol{u}_h-\boldsymbol{w}_h)\cdot\boldsymbol{\varphi}_h d\boldsymbol{x}\\
&=\frac{\mu}{\rho}\int_{\Omega} K^{-1}\boldsymbol{(u-w_h)}\cdot\boldsymbol{\varphi_h} d\boldsymbol{x}+\frac{\beta}{\rho}\int_{\Omega}\mathcal{D}\boldsymbol{f}_{\epsilon}(\boldsymbol{u}_H)(\boldsymbol{u}-\boldsymbol{w}_h)\cdot\boldsymbol{\varphi}_h d\boldsymbol{x}\\
&\quad-\frac{\beta}{\rho}\int_{\Omega}\mathcal{D}^2\boldsymbol{f}_{\epsilon}(\zeta_H)\cdot(\boldsymbol{u}-\boldsymbol{u}_H)^2\cdot\boldsymbol{\varphi}_h d\boldsymbol{x}+\int_{\Omega}\bigtriangledown(p-I_h(p))\cdot\boldsymbol{\varphi_h} d\boldsymbol{x}\\
&\quad-\frac{\beta}{\rho}\int_{\Omega}(|\boldsymbol{u}|_{\epsilon}-|\boldsymbol{u}|)\boldsymbol{u}\cdot\boldsymbol{\varphi}_h d\boldsymbol{x},\quad\forall\boldsymbol{\varphi_h}\in V_h.
\end{aligned}
\tag{4.46}
$$

Choose $\boldsymbol{\varphi_h}=\boldsymbol{u_h}-\boldsymbol{w_h}$ ($\boldsymbol{u_h}-\boldsymbol{w_h}\in V_h$ due to (4.13)), then (4.46) can be written as

$$
\tilde{T}=\sum_{i=1}^{5} T_i\ , \tag{4.47}
$$

with

$$\tilde{T} = \frac{\mu}{\rho}\int_\Omega K^{-1}(\boldsymbol{u_h} - \boldsymbol{w_h})\cdot(\boldsymbol{u}_h - \boldsymbol{w}_h)d\boldsymbol{x}$$
$$+ \frac{\beta}{\rho}\int_\Omega \mathcal{D}\boldsymbol{f}_\epsilon(\boldsymbol{u}_H)(\boldsymbol{u}_h - \boldsymbol{w}_h)\cdot(\boldsymbol{u}_h - \boldsymbol{w}_h)d\boldsymbol{x}, \tag{4.48}$$

$$T_1 = \frac{\mu}{\rho}\int_\Omega K^{-1}(\boldsymbol{u} - \boldsymbol{w}_h)\cdot(\boldsymbol{u}_h - \boldsymbol{w}_h)d\boldsymbol{x}\ , \tag{4.49}$$

$$T_2 = \frac{\beta}{\rho}\int_\Omega \mathcal{D}\boldsymbol{f}_\epsilon(\boldsymbol{u}_H)(\boldsymbol{u} - \boldsymbol{w}_h)\cdot(\boldsymbol{u}_h - \boldsymbol{w}_h)d\boldsymbol{x}\ , \tag{4.50}$$

$$T_3 = -\frac{\beta}{2!\rho}\int_\Omega \mathcal{D}^2\boldsymbol{f}_\epsilon(\zeta_H)\cdot(\boldsymbol{u} - \boldsymbol{u}_H)^2\cdot(\boldsymbol{u}_h - \boldsymbol{w}_h)d\boldsymbol{x}\ , \tag{4.51}$$

$$T_4 = \int_\Omega \bigtriangledown(p - I_h(p))\cdot(\boldsymbol{u}_h - \boldsymbol{w}_h)d\boldsymbol{x}\ , \tag{4.52}$$

$$T_5 = -\frac{\beta}{\rho}\int_\Omega (|\boldsymbol{u}|_\epsilon - |\boldsymbol{u}|)\boldsymbol{u}\cdot(\boldsymbol{u}_h - \boldsymbol{w}_h)d\boldsymbol{x}\ . \tag{4.53}$$

Thanks to Lemma 4.1, we estimate as follows:

$$\left|\tilde{T}\right| \geq \frac{\mu}{\rho}\lambda_s\|\boldsymbol{u}_h - \boldsymbol{w}_h\|^2_{[L^2(\Omega)]^d}, \tag{4.54}$$

$$|T_2| \leq \frac{\beta}{\rho}\gamma\|\boldsymbol{u} - \boldsymbol{w_h}\|_{[L^2(\Omega)]^d}\|\boldsymbol{u_h} - \boldsymbol{w_h}\|_{[L^2(\Omega)]^d}, \quad \gamma = \max_i \sum_{j=1}^2\left|\frac{\partial \boldsymbol{f}_{\epsilon,i}(\boldsymbol{u}_H)}{\partial u_j}\right|\ ,$$
$$\leq \frac{\beta}{\rho}(C_1 + \epsilon)\|\boldsymbol{u} - \boldsymbol{w_h}\|_{[L^2(\Omega)]^d}\|\boldsymbol{u_h} - \boldsymbol{w_h}\|_{[L^2(\Omega)]^d}\ , \tag{4.55}$$

$$|T_3| \leq \frac{\beta}{2!\rho}\int_\Omega\left|\sum_{k=1}^2\sum_{j=1}^2\sum_{i=1}^2\frac{\partial^2\boldsymbol{f}_{\epsilon,k}(\boldsymbol{\zeta}_H)}{\partial u_i\partial u_j}(u_i - u_{H,i})(u_j - u_{H,j})(u_{h,k} - w_{h,k})\right|dx\ ,$$
$$\leq \frac{\beta}{2!\rho}C\|\boldsymbol{u} - \boldsymbol{u}_H\|^2_{[L^2(\Omega)]^d}\|\boldsymbol{u}_h - \boldsymbol{w}_h\|_{[L^2(\Omega)]^d}\ , \tag{4.56}$$

$$|T_4| \leq \|\bigtriangledown(p - I_h(p))\|_{[L^2(\Omega)]^d}\|\boldsymbol{u_h} - \boldsymbol{w_h}\|_{[L^2(\Omega)]^d}\ , \tag{4.57}$$

$$|T_5| \leq \frac{\beta}{\rho}\epsilon\|\boldsymbol{u}\|_{[L^2(\Omega)]^d}\|\boldsymbol{u_h} - \boldsymbol{w_h}\|_{[L^2(\Omega)]^d}\ . \tag{4.58}$$

Substituting (4.54)-(4.58) in (4.47),

$$
\begin{aligned}
\frac{\mu}{\rho}\lambda_s\|\boldsymbol{u_h}-\boldsymbol{w_h}\|^2_{[L^2(\Omega)]^d} \leq\ & \frac{\mu}{\rho}\|K^{-1}\|_{L^\infty(\Omega)}\|\boldsymbol{u}-\boldsymbol{w_h}\|_{[L^2(\Omega)]^d}\|\boldsymbol{u_h}-\boldsymbol{w_h}\|_{[L^2(\Omega)]^d} \\
& + \frac{\beta}{\rho}(C_1+\epsilon)\|\boldsymbol{u}-\boldsymbol{w_h}\|_{[L^2(\Omega)]^d}\|\boldsymbol{u_h}-\boldsymbol{w_h}\|_{[L^2(\Omega)]^d} \\
& + \frac{\beta}{2!\rho}C\|\boldsymbol{u}-\boldsymbol{u}_H\|^2_{[L^2(\Omega)]^d}\|\boldsymbol{u}_h-\boldsymbol{w}_h\|_{[L^2(\Omega)]^d} \qquad (4.59) \\
& + \|\bigtriangledown(p-I_h(p))\|_{[L^2(\Omega)]^d}\|\boldsymbol{u_h}-\boldsymbol{w_h}\|_{[L^2(\Omega)]^d} \\
& + \frac{\beta}{\rho}\epsilon\|\boldsymbol{u}\|_{[L^2(\Omega)]^d}\|\boldsymbol{u_h}-\boldsymbol{w_h}\|_{[L^2(\Omega)]^d}\ . \qquad (4.60)
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\|\boldsymbol{u_h}-\boldsymbol{w_h}\|_{[L^2(\Omega)]^d} \leq\ & \frac{\|K^{-1}\|_{L^\infty(\Omega)}}{\lambda_s}\|\boldsymbol{u}-\boldsymbol{w_h}\|_{[L^2(\Omega)]^d} \\
& + \frac{\beta}{\mu\lambda_s}(C_1+\epsilon)\|\boldsymbol{u}-\boldsymbol{w_h}\|_{[L^2(\Omega)]^d} + \frac{\beta C}{2!\mu\lambda_s}\|\boldsymbol{u}-\boldsymbol{u}_H\|^2_{[L^2(\Omega)]^d} \\
& + \frac{\rho}{\mu\lambda_s}\|\bigtriangledown(p-I_h(p))\|_{[L^2(\Omega)]^d} + \frac{\beta}{\rho}\epsilon\|\boldsymbol{u}\|_{[L^2(\Omega)]^d}\ . \qquad (4.61)
\end{aligned}
$$

Error bound (4.44) follows from (4.61) ∎

Theorem 4.4 *Let $\mathcal{T}_h$ be a refinement of the triangulation $\mathcal{T}_H$ and satisfying* (3.1)*. Take $\boldsymbol{w}_h = \Pi_h(\boldsymbol{u})$ as defined in* (3.2)*. Then the approximate solution $\boldsymbol{u}_h$ satisfies the following order of convergence:*

$$\|\boldsymbol{u}-\boldsymbol{u_h}\|_{[L^2(\Omega)]^d} = O(\epsilon h + h + H^2 + \epsilon)\ . \qquad (4.62)$$

Proof. It follows immediately from (4.44) by using the stability properties in (3.5) and the order of convergence in (4.36). ∎

Remark 2 *For an appropriate choice of ϵ,* (4.62) *leads to a scaling $h = H^2$ between* *the coarse mesh size and fine mesh.*

4.3.2 Error estimates for pressure

Furthermore, we obtain an error bound for the pressure in the next Theorem.

Theorem 4.5 *Suppose the assumptions of Theorem 4.4 holds. If the solution p of the Problem (2.1)-(2.2) belongs to $W^{2,\frac{3}{2}}(\Omega)$, then there exists a constant C independent of h and H, such that*

$$\| \bigtriangledown (p - p_h)\|_{[L^{3/2}(\Omega)]^d} \leq C \left(\epsilon h + h + H^2 + \epsilon \right). \tag{4.63}$$

Proof. Subtracting (4.42) from (4.40), we get,

$$\sum_{T \in \mathcal{T}_h} \int_T \bigtriangledown (p - p_h) \cdot \boldsymbol{\varphi_h} d\boldsymbol{x} + \frac{\mu}{\rho} \int_\Omega K^{-1} (\boldsymbol{u} - \boldsymbol{u}_h) \cdot \boldsymbol{\varphi_h} d\boldsymbol{x} + \frac{\beta}{\rho} \int_\Omega \mathcal{D} \boldsymbol{f}_\epsilon (\boldsymbol{u}_H)(\boldsymbol{u} - \boldsymbol{u}_h) \cdot \boldsymbol{\varphi_h} d\boldsymbol{x}$$
$$= \frac{\beta}{2! \rho} \int_\Omega \mathcal{D}^2 \boldsymbol{f}_\epsilon (\boldsymbol{\zeta}_H)(\boldsymbol{u} - \boldsymbol{u}_H)^2 d\boldsymbol{x} + \frac{\beta}{\rho} \int_\Omega (|\boldsymbol{u}|_\epsilon - |\boldsymbol{u}|)\boldsymbol{u} \cdot \boldsymbol{\varphi}_h d\boldsymbol{x} \quad \forall \boldsymbol{\varphi}_h \in X_h. \tag{4.64}$$

Inserting $I_h(p)$ and we get,

$$\begin{aligned}
&\sum_{T \in \mathcal{T}_h} \int_T \bigtriangledown (I_h(p) - p_h) \cdot \boldsymbol{\varphi_h} d\boldsymbol{x} \\
&= -\frac{\mu}{\rho} \int_\Omega K^{-1} (\boldsymbol{u} - \boldsymbol{u}_h) \cdot \boldsymbol{\varphi_h} d\boldsymbol{x} - \frac{\beta}{\rho} \int_\Omega \mathcal{D} \boldsymbol{f}_\epsilon (\boldsymbol{u}_H)(\boldsymbol{u} - \boldsymbol{u}_h) \cdot \boldsymbol{\varphi_h} d\boldsymbol{x} \\
&\quad + \frac{\beta}{2! \rho} \int_\Omega \mathcal{D}^2 \boldsymbol{f}_\epsilon (\boldsymbol{\zeta}_H)(\boldsymbol{u} - \boldsymbol{u}_H)^2 \cdot \boldsymbol{\varphi_h} d\boldsymbol{x} + \frac{\beta}{\rho} \int_\Omega (|\boldsymbol{u}|_\epsilon - |\boldsymbol{u}|)\boldsymbol{u} \cdot \boldsymbol{\varphi}_h d\boldsymbol{x} \\
&\quad - \sum_{T \in \mathcal{T}_h} \int_T \bigtriangledown (p - I_h(p)) \cdot \boldsymbol{\varphi_h} d\boldsymbol{x}, \quad \forall \boldsymbol{\varphi_h} \in X_h.
\end{aligned} \tag{4.65}$$

It follows that

$$
\begin{aligned}
&\left|\sum_{T\in\mathcal{T}_h}\int_T \bigtriangledown(I_h(p)-p_h)\cdot\boldsymbol{\varphi_h}d\boldsymbol{x}\right| \\
&\leq \frac{\mu}{\rho}\|K^{-1}\|_{L^\infty(\Omega)}|\Omega|^{\frac{1}{3}}\|\boldsymbol{u}-\boldsymbol{u}_h\|_{[L^2(\Omega)]^d}\|\boldsymbol{\varphi_h}\|_{[L^3(\Omega)]^d} \\
&\quad+\frac{\beta}{\rho}\max_i\sum_{j=1}^{2}\left|\frac{\partial \boldsymbol{f}_{\epsilon,i}(\boldsymbol{u}_H)}{\partial u_j}\right||\Omega|^{\frac{1}{3}}\|\boldsymbol{u}-\boldsymbol{u}_h\|_{[L^2(\Omega)]^d}\|\boldsymbol{\varphi_h}\|_{[L^3(\Omega)]^d} \\
&\quad+\frac{\beta}{2!\rho}\left|\sum_{k=1}^{2}\sum_{j=1}^{2}\sum_{i=1}^{2}\frac{\partial^2 \boldsymbol{f}_{\epsilon,k}(\boldsymbol{\zeta}_H)}{\partial u_i\partial u_j}\right|\|(\boldsymbol{u}-\boldsymbol{u}_H)^2\|_{[L^2(\Omega)]^d}|\Omega|^{\frac{1}{3}}\|\boldsymbol{\varphi_h}\|_{[L^3(\Omega)]^d} \\
&\quad+\epsilon|\Omega|^{\frac{1}{3}}\|\boldsymbol{u}\|_{[L^2(\Omega)]^d}\|\boldsymbol{\varphi}_h\|_{[L^3(\Omega)]^d}+\|\bigtriangledown(p-I_h(p))\|_{[L^{\frac{3}{2}}(\Omega)]^d}\|\boldsymbol{\varphi_h}\|_{[L^3(\Omega)]^d}\ .
\end{aligned}
$$

Applying the bounds for the partial derivatives in Lemma 4.1 and dividing through by $\|\boldsymbol{\varphi_h}\|_{[L^3(\Omega)]^d}$ for $\boldsymbol{\varphi_h}\in X_h\setminus\{0\}$, we obtain

$$
\begin{aligned}
&\left|\frac{1}{\|\boldsymbol{\varphi_h}\|_{[L^3(\Omega)]^d}}\sum_{T\in\mathcal{T}_h}\int_T \bigtriangledown(I_h(p)-p_h)\cdot\boldsymbol{\varphi_h}d\boldsymbol{x}\right| \\
&\leq \frac{\mu}{\rho}\|K^{-1}\|_{L^\infty(\Omega)}|\Omega|^{\frac{1}{3}}\|\boldsymbol{u}-\boldsymbol{u}_h\|_{[L^2(\Omega)]^d} \\
&\quad+\frac{\beta}{\rho}(C_1+\epsilon)|\Omega|^{\frac{1}{3}}\|\boldsymbol{u}-\boldsymbol{u}_h\|_{[L^2(\Omega)]^d}+\frac{C\beta}{2!\rho}|\Omega|^{\frac{1}{3}}\|\boldsymbol{u}-\boldsymbol{u}_H\|^2_{[L^2(\Omega)]^d}+\epsilon|\Omega|^{\frac{1}{3}}\|\boldsymbol{u}\|_{[L^2(\Omega)]^d} \\
&\quad+\|\bigtriangledown(p-I_h(p))\|_{[L^{\frac{3}{2}}(\Omega)]^d}\ .
\end{aligned}
$$

In view of the discrete infsup condition (4.24), we get

$$
\begin{aligned}
\|\bigtriangledown(I_h(p)-p_h)\|_{[L^{\frac{3}{2}}(\Omega)]^d} &\leq \left(\frac{\mu}{\rho}|\Omega|^{\frac{1}{3}}\|K^{-1}\|_{L^\infty(\Omega)}+\frac{\beta}{\rho}(C_1+\epsilon)|\Omega|^{\frac{1}{3}}\right)\|\boldsymbol{u}-\boldsymbol{u}_h\|_{[L^2(\Omega)]^d} \\
&\quad+\frac{C\beta}{2!\rho}|\Omega|^{\frac{1}{3}}\|\boldsymbol{u}-\boldsymbol{u}_H\|^2_{[L^2(\Omega)]^d}+\epsilon|\Omega|^{\frac{1}{3}}\|\boldsymbol{u}\|_{[L^2(\Omega)]^d} \\
&\quad+\|\bigtriangledown(p-I_h(p)\|_{[L^{\frac{3}{2}}(\Omega)]^d}. \qquad (4.66)
\end{aligned}
$$

Owing to the stability property of $I_h(p)$ in (3.5) and the error estimate in (4.44), we deduce (4.63). $\blacksquare$

CHAPTER 5

NUMERICAL IMPLEMENTATION

5.1 Linear and nonlinear systems of equations

In this section, we present the solutions of both the nonlinear system associated with the coarse mesh and the linear system associated with the fine mesh.

As usual, we choose bases $X_H = span\,\{\boldsymbol{\varphi}_i\}_{i=1}^{n_1}$, $M_H = span\,\{\psi_i\}_{i=1}^{m_1}$,

$X_h = span\,\{\boldsymbol{\Phi}_i\}_{i=1}^{n_2}$ and $M_h = span\,\{\Psi_i\}_{i=1}^{m_2}$. Then for any $\boldsymbol{u}_H \in X_H$, $p_H \in M_H$, and $\boldsymbol{u}_h \in X_h$, $p_h \in M_h$, can be expressed as

$$\boldsymbol{u}_H = \sum_{i=1}^{n_1} \tilde{u_1}^i \boldsymbol{\varphi}_i \qquad p_H = \sum_{i=1}^{m_1} \tilde{p_1}^i \psi_i \ , \tag{5.1}$$

$$\boldsymbol{u}_h = \sum_{i=1}^{n_2} \tilde{u_2}^i \boldsymbol{\Phi}_i \qquad p_h = \sum_{i=1}^{m_2} \tilde{p_2}^i \Psi_i \ , \tag{5.2}$$

Now substituting (5.1)) and (5.2) into equation (4.10)-(4.11), (4.12)-(4.13) respectively results to the systems of equation given below.

Find $\tilde{u}_1 \in \mathbb{R}^{n_1}$, $\tilde{p}_1 \in \mathbb{R}^{m_1}$ such that,

$$\begin{bmatrix} D(\tilde{u}_1) & B^T \\ \\ B & 0 \end{bmatrix} \begin{bmatrix} \tilde{u}_1 \\ \\ \tilde{p}_1 \end{bmatrix} = \begin{bmatrix} G_1 \\ \\ G_2 \end{bmatrix}, \tag{5.3}$$

where

$$\tilde{u_1} = [\tilde{u_1}^1, \tilde{u_1}^2, \cdots, \tilde{u_1}^{n_1}]^T \quad \tilde{p_1} = [\tilde{p_1}^1, \tilde{p_1}^2, \cdots, \tilde{p_1}^{m_1}]^T.$$

Find $\tilde{u}_2 \in \mathbb{R}^{n_2}$, $\tilde{p}_2 \in \mathbb{R}^{m_2}$ such that,

$$\begin{bmatrix} \bar{D}(\tilde{u}_1) & \bar{B}^T \\ \\ \bar{B} & 0 \end{bmatrix} \begin{bmatrix} \tilde{u}_2 \\ \\ \tilde{p}_2 \end{bmatrix} = \begin{bmatrix} \bar{G}_1 \\ \\ \bar{G}_2 \end{bmatrix}, \tag{5.4}$$

where

$$\tilde{u_2} = [\tilde{u_2}^1, \tilde{u_2}^2, \cdots, \tilde{u_2}^{n_2}]^T \qquad \tilde{p_2} = [\tilde{p_2}^1, \tilde{p_2}^2, \cdots, \tilde{p_2}^{m_2}]^T.$$

$D(\tilde{u}_1) \in \mathbb{R}^{n_1 \times n_1}$ is a diagonal matrix and $D(\tilde{u}_1)\tilde{u}_1$ is a matrix corresponding to:

$$\frac{\mu}{\rho} \int_\Omega K^{-1} \boldsymbol{u}_H \cdot \boldsymbol{\varphi}_H d\boldsymbol{x} + \frac{\beta}{\rho} \int_\Omega |\boldsymbol{u}_H| \boldsymbol{u}_H \cdot \boldsymbol{\varphi}_H d\boldsymbol{x} \ .$$

$\bar{D}(\tilde{u}_1) \in \mathbb{R}^{n_2 \times n_2}$ is a diagonal matrix and $\bar{D}(\tilde{u}_1)\tilde{u}_2$ is a matrix corresponding to:

$$\frac{\mu}{\rho} \int_\Omega (K^{-1} \boldsymbol{u}_h) \cdot \boldsymbol{\varphi}_h dx + \frac{\beta}{\rho} \int_\Omega \mathcal{D}\boldsymbol{f}_\epsilon(\boldsymbol{u}_H) \boldsymbol{u}_h \cdot \boldsymbol{\varphi}_h d\boldsymbol{x}. \tag{5.5}$$

$B \in \mathbb{R}^{m_1 \times n_1}$, $B^T \in \mathbb{R}^{n_1 \times m_1}$ are matrices corresponding to:

$$\int_\Omega \nabla q_H \cdot \boldsymbol{u}_H d\boldsymbol{x}, \quad \int_\Omega \nabla p_H \cdot \boldsymbol{\varphi}_H d\boldsymbol{x} \ ,$$

respectively. $\bar{B} \in \mathbb{R}^{m_2 \times n_2}$, $\bar{B}^T \in \mathbb{R}^{n_2 \times m_2}$ are matrices corresponding to,

$$\int_\Omega \nabla q_h \cdot \boldsymbol{u}_h d\boldsymbol{x}, \quad \int_\Omega \nabla p_h \cdot \boldsymbol{\varphi}_h d\boldsymbol{x} \ ,$$

respectively. $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$ are column vectors corresponding to:

$$\int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi}_H dx, \quad \text{and} \quad -\int_\Omega b q_H \boldsymbol{d} x + \int_\Gamma g q_H \boldsymbol{d}\sigma \ ,$$

respectively. $\bar{\boldsymbol{G}}_1$ and $\bar{\boldsymbol{G}}_2$ are column vectors corresponding to:
$\displaystyle\int_\Omega \boldsymbol{f}\boldsymbol{\varphi}_h dx - \frac{\beta}{\rho}\int_\Omega |\boldsymbol{u}_{\boldsymbol{H}}|_\epsilon \boldsymbol{u}_{\boldsymbol{H}} \cdot \boldsymbol{\varphi}_h d\boldsymbol{x} + \frac{\beta}{\rho}\int_\Omega \mathcal{D}\boldsymbol{f}_\epsilon(\boldsymbol{u}_H)\boldsymbol{u}_H \cdot \boldsymbol{\varphi}_h d\boldsymbol{x}$ and $\displaystyle -\int_\Omega b q_h \boldsymbol{d} x +$ $\displaystyle\int_\Gamma g q_h \boldsymbol{d}\sigma$, respectively.

5.2 Numerical results

In this section, we present some numerical results to validate the effectiveness of the two-level algorithm. The purpose of this section is firstly, to illustrate the computational efficiency of the two-level method, and secondly, to verify the theoretical rates of convergence developed in Chapter 5. We solved the nonlinear system (5.3) on the coarse mesh by Newton's method [37, 49] while the linear systems (5.4) on the fine mesh was solved by MINRES [50]. We will consider two test problems.

5.2.1 Problem with known solution

Here, we consider an example taken from [57, 61]. $\Omega = (-1,1) \times (-1,1)$, $g = 0$ on $\partial\Omega$, and

$$\boldsymbol{f}(x,y) = \begin{bmatrix} 2y(1-x^2)\left(1+2\beta\sqrt{y^2(1-x^2)^2+x^2(1-y^2)^2}\right)+3x^2 \\ -2x(1-y^2)\left(1+2\beta\sqrt{y^2(1-x^2)^2+x^2(1-y^2)^2}\right)+3y^2 \end{bmatrix}.$$

Then the exact velocity and pressure are given as follows:

$$\boldsymbol{u}(x,y) = \left(2y(1-x^2), -2x(1-y^2)\right)^T, \quad p(x,y) = x^3+y^3.$$

Computational efficiency

Here we compare the simulation time for the standard one-level method with the simulation time for the two-level method. For this experiment, K is assumed to be I_{2X2}, μ and ρ were taken to be 1 . The non-Darcy parameter β is assumed to be 0.1 and $\epsilon = 10^{-3}$. In Table 5.1, the degrees of freedom of the velocity field and pressure are tabulated, it shows the division of huge nonlinear systems of equation into a smaller nonlinear system of the equations on the coarse mesh followed by a large linear system on the fine mesh.

The simulation time for different mesh sizes for example (5.2.1) are presented in Tables 5.2 and 5.3. The two-level method appears to be much more cheaper than the one-level method for smaller values of h without compromising accuracy.

Rate of convergence

The L^2-errors of the velocity, and its rate of convergence with respect to h between the coarse and fine mesh are presented in Table 5.4. The order of convergence agrees with the result in (4.44).

5.2.2 Parameter Studies

In this subsection, we investigate the effect of the nonlinearity. The simulation time is recorded for different values of the Forchheimer coefficient β. The results are tabulated in Table 5.5. The velocity and pressure profiles for different values of β are displayed in Figures 6.1, 6.2 and 6.3.

Table 5.1: Degrees of freedoms comparison of one-level and two-level method

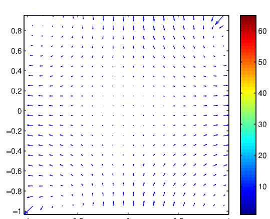
H	h	DoFu_H	DoFp_H	DoFu_h	DoFp_h	DoF coarse	DoF fine
	1/16			4096	1089		5185
1/4	1/16	256	81	4096	1089	337	5185
	1/36			20736	5329		26065
1/6	1/36	576	169	20736	5329	745	26065
	1/100			160000	40401		200401
1/10	1/100	1600	441	160000	40401	2041	200401
	1/144			331776	83521		415297
1/12	1/144	2304	625	331776	83521	2929	415297

Table 5.2: Error and CPU time comparison of one-level and two-level methods

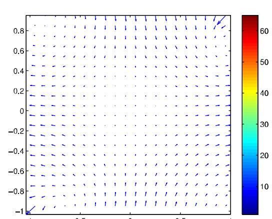
H	h	$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^2(\Omega)}$	CPU(sec.)
	1/16	0.20263	2.31759
1/4	1/16	0.20095	2.29551
	1/36	0.09346	11.7719
1/6	1/36	0.09061	2.66802
	1/64	0.05645	62.4759
1/8	1/64	0.05182	8.98689
	1/100	0.04078	266.195
1/10	1/100	0.03415	28.0365
	1/144	0.03334	1005.27
1/12	1/144	0.02488	78.2143

Table 5.3: Error and CPU time comparison two-level methods with fixed h

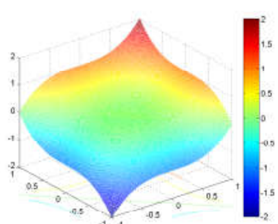
H	h	$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^2(\Omega)}$	CPU(sec.)
1/4	1/144	0.02594	58.05769
1/8	1/144	0.02506	62.71531
1/16	1/144	0.02483	83.67437
1/32	1/144	0.02484	160.12118
1/64	1/144	0.02479	494.37599



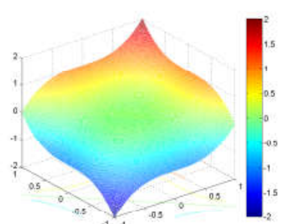
(a) One level velocity



(b) Two level velocity



(c) One level pressure



(d) Two level pressure

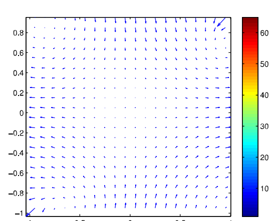
Figure 5.1: One level versus Two level profiles for $\beta = 15$

Table 5.4: Two-level methods: Rate of convergence with respect to h

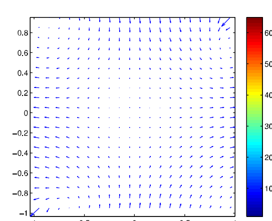
H	h	$\|\boldsymbol{u} - \boldsymbol{u_h}\|_{L^2(\Omega)}$	L^2-order
1/4	1/16	0.20095	—
1/6	1/36	0.90606	0.98215
1/8	1/64	0.05182	0.97124
1/10	1/100	0.03416	0.93395
1/12	1/144	0.02489	0.86810

Table 5.5: Two-level methods: Effect of Forchheimer coefficient

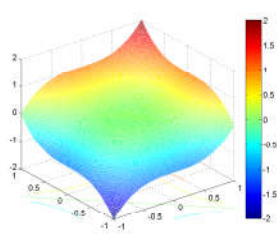
β	H	h	CPU(sec.)
15		1/100	520.2279
	1/10	1/100	48.0028
10		1/100	433.0770
	1/10	1/100	45.20978
8		1/100	431.5467
	1/10	1/100	36.0425
5		1/100	349.7195
	1/10	1/100	30.8889
0.1		1/100	266.1951
	1/10	1/100	28.0365



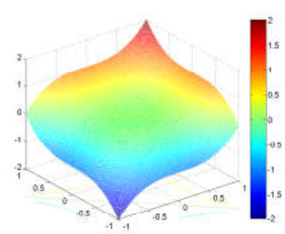
(a) One level velocity



(b) Two level velocity

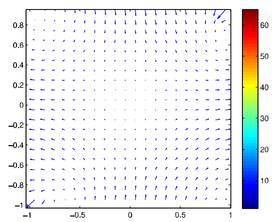


(c) One level pressure

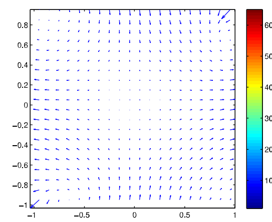


(d) Two level pressure

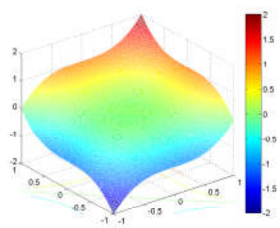
Figure 5.2: One level versus Two level profiles for $\beta = 10$



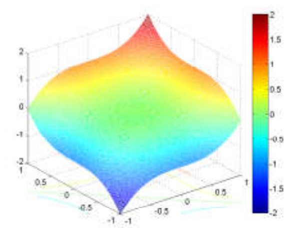
(a) One level velocity



(b) Two level velocity



(c) One level pressure



(d) Two level pressure

Figure 5.3: One level versus Two level profiles for $\beta = 8$

5.2.3 Five-spot problem

In this subsection, We consider the five-spot problem which is a well-known test-problem for numerical techniques in reservoir simulation. We consider an isotropic and homogeneous permeability $K = 1$ for all $x \in \Omega = [-1, 1] \times [-1, 1]$. We place one injection well at the center and four production wells at the corners and no-flow conditions at the boundaries [1] . The pressure profile of example (5.2.3) is shown in Figure5.4 and Figure 5.5.The profile is as expected of a five spot problem, high pressure distributions are observed at the corners of the square (production wells).

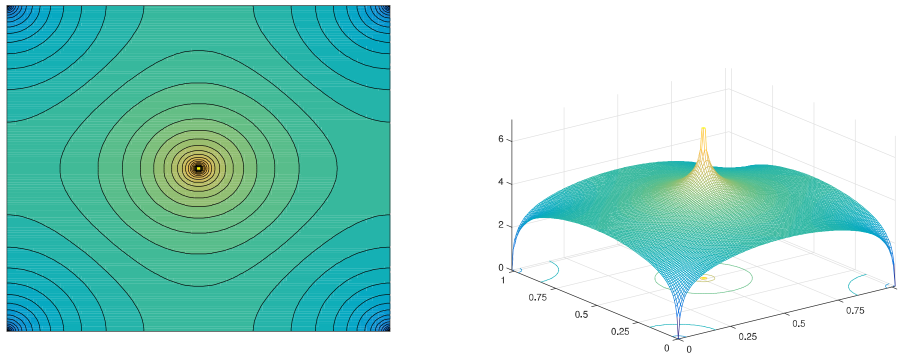


Figure 5.4: Pressure profile for the five spot problem left:pressure streamlines, right:3D pressure surface.

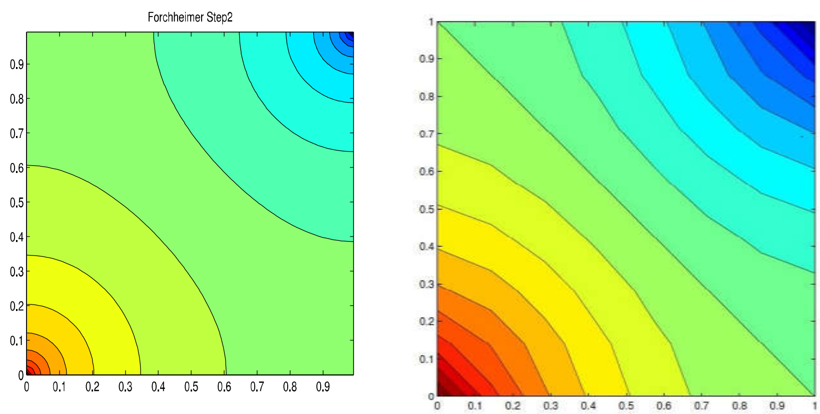


Figure 5.5: Pressure profile for the five spot problem left: Two-level method, right: Aarnes et al

5.2.4 Results for Generalized Darcy-Forchheimer

The mixed finite element approximation for the generalized Darcy-Forchheimer equation is implemented. The errors and order of convergence for different values of m are tabulated in Tables 5.6 and 5.7.

Table 5.6: Rate of convergence when $m = 3/2$

h	$\|\boldsymbol{u} - u_h\|_{L^{\frac{3}{2}}(\Omega)}$	$L^{\frac{3}{2}}$-order	$\| \bigtriangledown (p - p_h)\|_{L^{\frac{5}{3}}(\Omega)}$	$L^{\frac{5}{3}}$-order
1/4	0.5184	—-	0.1547	—-
1/8	0.2681	0.9512	0.0570	1.4420
1/16	0.1374	0.9648	0.2008	1.4541
1/32	0.0726	0.9194	0.0078	1.4119
1/64	0.0427	0.7659	0.0036	1.1122

Table 5.7: Rate of convergence when $m = 6/5$

h	$\|\boldsymbol{u} - u_h\|_{L^{\frac{6}{5}}(\Omega)}$	$L^{\frac{6}{5}}$-order	$\| \bigtriangledown (p - p_h)\|_{L^{\frac{11}{6}}(\Omega)}$	$L^{\frac{11}{6}}$-order
1/4	0.6965	—-	0.1760	—-
1/8	0.3611	0.9476	0.0709	1.3120
1/16	0.1848	0.9661	0.0283	1.3243
1/32	0.0970	0.9304	0.0113	1.3215
1/64	0.0558	0.7971	0.0049	1.2214

CHAPTER 6

CONCLUSION AND FUTURE WORK

In this dissertation, we proved the well-posedness of the generalized Darcy-Forchheimer model, modelling a single phase steady non-Darcy flow in two or three dimensional porous media. We proposed a mixed finite element scheme, established the well-posedness of the scheme and also implemented it. In addition, we formulated and implemented a two-level method mixed finite algorithm for Darcy-Forchheimer equation . In order to construct the two-level algorithm, the nonlinear term was modified to allow up to second order Taylor series expansion. Numerical experiments for the two-level algorithm with piecewise constant velocities and continuous piecewise linear pressures were carried out. The numerical results verified the theoretical error estimates, both with respect to the coarse mesh size, H, and the fine mesh size, h. We proved that the error estimates of the two-level mixed finite element algorithm are of order $O(\epsilon h + h + H^2 + \epsilon)$ for velocity and pressure respectively. The two-level method significantly decreases the computational time of the standard one-level method while maintaining accuracy. The method provides a reliable solution for highly nonlinear flows. The direction for future works include:

- The use of higher-order Taylor series expansion of the nonlinear term to improve the convergence rate.
- The implementation of two-level method for the general case $m \in (1, 2]$.
- The analysis and implementation of two-level method for multi-phase non Darcy flows in porous media.

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